

# DECOMPOSITIONS OF SURFACE FLOWS

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**ABSTRACT.** Let  $v$  be a flow with non-degenerate singular points on a compact surface. We describe properties of border points of  $v$  and characterize the necessary and sufficient condition for the closure of union of closed orbit corresponding with the non-wandering set. Moreover, each connected component of the complement of the “saddle connection diagram” is either an open disk, an open annulus, a torus, a Klein bottle, an open Möbius band, or an open essential subset. In addition, though the set of topological equivalent classes of minimal flows (resp. Denjoy flows) on a torus is uncountable, a flow with non-degenerate singular points and with at most finitely many limit cycles but without quasi-minimal sets on a compact surface can be reconstructed by finite data.

## 1. INTRODUCTION

The Poincaré-Bendixson theorem is generalized in many ways. For instance, one of them states a following statement (see for example [11]): The  $\omega$ -limit set of an orbit of a flow with finitely many fixed points on a compact surface is either a closed orbit, an attracting limit circuit, or a quasi-minimal set.

In [12], M. Peixoto has shown that a vector field  $v \in \chi^r(S)$  is structurally stable if and only if  $v \in \Sigma^r(S)$ , where  $\chi^r(S)$  ( $1 \leq r \leq \infty$ ) is the set of  $C^r$ -vector fields (with the  $C^r$ -topology) on an orientable closed connected surface  $S$  and  $\Sigma^r(S)$  is the subset of  $\chi^r(S)$  formed by the Morse-Smale  $C^r$ -vector fields. Moreover,  $\Sigma^r(S)$  is open and dense in  $\chi^r(S)$ . Recall that a  $C^r$  vector field ( $r \geq 1$ ) on  $S$  with finitely many singular points is Morse-Smale if 1) each singular point is hyperbolic (i.e. a  $(\partial)$ -sink, a  $(\partial)$ -source, or a saddle), 2) each periodic orbit is a hyperbolic limit cycle, 3) there are no heteroclinic multi-saddle separatrices, and 4) the  $\omega$ -limit (resp.  $\alpha$ -limit) set of a point is a closed orbit. When  $S$  is non-orientable, Pugh’s  $C^1$ -Closing Lemma implies [13, 14] that  $\Sigma^1(S)$  is dense and the Peixoto’s work holds for the case that the non-orientable genus of  $S$  is less than 5 [3, 7]. On Hamiltonian vector fields, a Hamiltonian vector field  $v \in \mathcal{H}^r(S)$  is structurally stable in  $\mathcal{H}^r(S)$  if and only if  $v \in \mathcal{H}_*^r(S)$ , and that  $\mathcal{H}_*^r(S)$  is open dense in  $\mathcal{H}^r(S)$  [5], where  $\mathcal{H}^r(S)$  is the set of Hamiltonian  $C^r$  vector field on an orientable closed connected surface  $S$  and  $\mathcal{H}_*^r(S)$  is the set of regular Hamiltonian  $C^r$  vector field each of whose saddle connection is self-connected.

In this paper, we describe properties of border points of a flow  $v$  with non-degenerate singular points on a compact surface and characterize the necessary and sufficient condition for the closure of union of closed orbit corresponding with the non-wandering set. Moreover, each connected component of the complement of

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the “saddle connection diagram” is either an open disk, an open annulus, a torus, a Klein bottle, an open Möbius band, or an open essential subset. In addition, though the set of topological equivalent classes of minimal flows (resp. Denjoy flows) on a torus is uncountable, a flow with non-degenerate singular points and with at most finitely many limit cycles but without quasi-minimal sets on a compact surface can be reconstructed by finite data.

## 2. PRELIMINARIES

**2.1. Notions of dynamical systems.** By a surface, we mean a compact two dimensional manifold, which needs not orientable. A flow is a continuous  $\mathbb{R}$ -action on a surface. Let  $v : \mathbb{R} \times S \rightarrow S$  be a flow on a surface  $S$ . For  $t \in \mathbb{R}$ , define  $v_t : S \rightarrow S$  by  $v_t := v(t, \cdot)$ . For a point  $x$  of  $S$ , we denote by  $O(x)$  the orbit of  $x$ . Recall that a point  $x$  of  $S$  is singular if  $x = v_t(x)$  for any  $t \in \mathbb{R}$ , is regular if  $x$  is not singular, and is periodic if there is positive number  $T > 0$  such that  $x = v_T(x)$  and  $x \neq v_t(x)$  for any  $t \in (0, T)$ . An orbit is closed if it is singular or periodic. Denote by  $\text{Sing}(v)$  (resp.  $\text{Per}(v)$ ,  $\text{Cl}(v)$ ) the set of singular (resp. periodic, closed) points. Recall that the  $\omega$ -limit (resp.  $\alpha$ -limit) set of a point  $x$  is  $\omega(x) := \bigcap_{n \in \mathbb{R}} \overline{\{v_t(x) \mid t > n\}}$  (resp.  $\alpha(x) := \bigcap_{n \in \mathbb{R}} \overline{\{v_t(x) \mid t < -n\}}$ ), where the closure of a subset  $A$  is denoted by  $\overline{A}$ . A point is wandering if there are a neighbourhood  $U$  of it and positive number  $N$  such that  $v_t(U) \cap U = \emptyset$  for any  $t > N$ . A point is non-wandering if it is not wandering (i.e. for any neighbourhood  $U$  of it and for any positive number  $N$ , there is  $t \in \mathbb{R}$  with  $|t| > N$  such that  $v_t(U) \cap U \neq \emptyset$ ). Denote by  $\Omega$  the set of non-wandering points, called the non-wandering set. For an orbit  $O$ , define  $\omega(O) := \omega(x)$  and  $\alpha(O) := \alpha(x)$  for some point  $x \in O$ . Note an  $\omega$ -limit (resp.  $\alpha$ -limit) set of an orbit is independent of the choice of a point in the orbit. A subset is saturated (or invariant) if it is a union of orbits. The saturation of a subset is the union of orbits intersecting it. Denote by  $\text{Sat}_v(A)$  the saturation of a subset  $A$ . A point  $x$  of  $S$  is recurrent (resp. weakly recurrent) if  $x \in \omega(x) \cap \alpha(x)$  (resp.  $x \in \omega(x) \cup \alpha(x)$ ). A quasi-minimal set is an orbit closure of a weakly recurrent orbit. It's known that the total number of quasi-minimal sets for  $v$  cannot exceed  $g$  if  $M$  is an orientable surface of genus  $g$  [6], and  $\frac{p-1}{2}$  if  $M$  is a non-orientable surface of genus  $p$  [8]. Therefore the closure  $\overline{\text{LD} \sqcup \text{E}}$  is a finite union of quasi-minimal sets.

**2.2. Types of singular points.** A singular point  $x$  on (resp. outside of) the boundary  $\partial S$  is a  $\partial$ -sink (resp. sink) if there is a neighborhood  $U$  of  $x$  such that  $\omega(y) = \{x\}$  for any  $y \in U$ , is a  $\partial$ -source (resp. source) if there is a neighborhood  $U$  of  $x$  such that  $\alpha(y) = \{x\}$  for any  $y \in U$ , is a center if there is a saturated neighborhood  $U$  of  $x$  such that  $U - \{x\}$  consists of periodic orbits. A separatrix is a regular orbit whose  $\alpha$ -limit or  $\omega$ -limit set is a singular point. A separatrix is connecting if each of the  $\omega$ -limit set of it and the  $\alpha$ -limit sets of it is a singular point. A  $\partial$ - $k$ -saddle (resp.  $k$ -saddle) is an isolated singular point on (resp. outside of)  $\partial S$  with exactly  $(2k + 2)$ -separatrices, counted with multiplicity, where  $\partial S$  is the boundary of a surface  $S$ . A multi-saddle is a  $k$ -saddle or a  $\partial$ -( $k/2$ )-saddle for some  $k \in \mathbb{Z}_{\geq 0}$ . A 1-saddle is topologically an ordinary saddle.

**2.3. (Quasi-)Regularity.** A flow is quasi-regular (resp. regular) if each singular point is either a center, a multi-saddle (resp. a saddle, a  $\partial$ -saddle), a sink, a  $\partial$ -sink, a source, or a  $\partial$ -source. Note that a non-wandering flow with finitely many singular

points on a compact surface is quasi-regular [2]. Conversely, a quasi-regular flow on a compact surface has finitely many singular points.

**2.4. Topological properties of orbits.** An orbit is proper if it is embedded (i.e. there is a neighborhood of it where the orbit is closed), locally dense if the closure of it has nonempty interior, and exceptional if it is neither proper nor locally dense. A point is proper (resp. locally dense, exceptional) if so is its orbit. Denote by LD (resp. E, P) the union of locally dense orbits (resp. exceptional orbits, non-closed proper orbits). Note that P is the complement of the set of weakly recurrent points and that the union  $LD \sqcup E$  is the union of non-closed weakly recurrent orbits. By the definitions, we have a decomposition  $S = \text{Sing}(v) \sqcup \text{Per}(v) \sqcup P \sqcup LD \sqcup E$ , where  $\sqcup$  is the disjoint union symbol. For a closed invariant set  $\gamma$ , define the stable manifold  $W^s(\gamma) := \{y \in S \mid \omega(y) \subseteq \gamma\}$  and the unstable manifold  $W^u(\gamma) := \{y \in S \mid \alpha(y) \subseteq \gamma\}$ .

**2.5. Topological notions.** A quasi-minimal set is exceptional (resp. locally dense) if it is a closure of an orbit intersecting E (resp. LD). Recall that the (orbit) class  $\hat{O}$  of an orbit  $O$  is the union of orbits each of whose orbit closure corresponds with  $\overline{O}$  (i.e.  $\hat{O} = \{y \in S \mid \overline{O(y)} = \overline{O}\}$ ). Denote by  $\partial A := \overline{A} - \text{int}A$  the boundary of a subset  $A \subseteq S$ . Define the border  $\delta A := A - \text{int}A$  of a subset  $A \subseteq S$ . Denote by  $\sigma A := \overline{A} - A$  the shell of a subset  $A \subseteq S$ . Then  $\partial A = \delta A \sqcup \sigma A$ . A 1-dimensional cell complex is essential if it is not null homologous with respect to the relative homology group  $H_1(S, \partial S; \mathbb{Z})$ . Notice that a 1-dimensional cell complex  $\gamma$  is essential in a compact surface  $S$  if and only if  $\gamma$  is not null homotopic in  $S^*$ , where  $S^*$  is the resulting closed surface from  $S$  by collapsing all boundaries into singletons. An open subset of  $S$  is essential if it contains no essential 1-dimensional cell complex. A singular point is called a compressed center if for any neighborhood  $U$  of it there is an open neighborhood  $V \subset U$  of it whose boundary is a periodic orbit.

**2.6. Transversality.** Recall a curve is a continuous mapping  $C : I \rightarrow S$  where  $I$  is a non-degenerate connected closed subset of  $\mathbb{S}^1$ . A curve is simple if it is injective. We also denote by  $C$  the image of a curve  $C$ . Denote by  $\partial C := C(\partial I)$  the boundary of a curve  $C$ , where  $\partial I$  is a boundary of  $I \subset \mathbb{S}^1$ . A curve  $C$  is transverse to  $v$  at a point  $p$  if there are an orbit arc  $\gamma$  containing  $p$  and a small open disk  $U$  centered at  $p$  such that  $C \cap \gamma = \{p\}$ ,  $\gamma - \{p\} \subset U \setminus C$ , and  $U \setminus C$  is two open disks each of which intersects  $\gamma$ . A simple curve  $C$  is transverse to  $v$  if it is transverse to  $v$  at each point in  $C - \partial C$ . A simple curve  $C$  transverse to  $v$  is called a transverse arc. A simple closed curve is a closed transversal if it transverses to  $v$ .

**2.7. One-sided loops and two sided loops.** By a loop, we mean a simple closed curve in a surface. A loop is one-sided if there is a small neighborhood of it which is a Möbius band. A loop is two-sided if there is a small annular neighborhood of it.

**2.8. Multi-saddle connection diagrams.** A connecting separatrix is a multi-saddle separatrix if each of the  $\alpha$ -limit and  $\omega$ -limit set of it is a multi-saddle. The multi-saddle connection diagram is the union of multi-saddles and multi-saddle separatrices. A multi-saddle connection is a connected component of the multi-saddle connection diagram. Note that a multi-saddle connection is also called a poly-cycle. A multi-saddle separatrix is heteroclinic if it connects either distinct multi-saddles off boundaries, or a multi-saddle on a boundary and a multi-saddle

off boundaries, or multi-saddles on distinct boundary components. A connecting separatrix is homoclinic if either the  $\omega$ -limit set and  $\alpha$ -limit set of it correspond or it connects multi-saddles on the same boundary component.

**2.9. Ss-multi-saddle connection diagrams.** An invariant subset is an ss-component if it is either a sink, a  $\partial$ -sink, a source, a  $\partial$ -source, a limit circuit, or an exceptional quasi-minimal set. A separatrix is an ss-separatrix if it connects a multi-saddle and an ss-component. In other words, a separatrix whose  $\omega$ -limit set is a multi-saddle is an ss-separatrix if the  $\alpha$ -limit set of it is an ss-component. A separatrix whose  $\alpha$ -limit set is a multi-saddle is an ss-separatrix if the  $\omega$ -limit set of it is an ss-component. The ss-multi-saddle connection diagram  $D_{ss}$  is the union of multi-saddles, multi-saddle separatrices, ss-separatrices, and ss-components. An ss-multi-saddle connection is a connected component of the ss-multi-saddle connection diagram.

**2.10. Self-connectedness.** A (multi-)saddle connection is a connected component of the (multi-)saddle connection diagram. A (multi-)saddle connection which contains no boundaries of  $S$  is self-connected if it is the union of a (multi-)saddle and homoclinic connecting separatrices. A (multi-)saddle connection which contains a boundary of  $S$  is self-connected if each separatrix contained in it connects points on the boundary component. The (multi-)saddle connection diagram is self-connected if so is each connected component.

**2.11. Circuits.** By a cycle, we mean a periodic orbit. By a non-trivial circuit, we mean either a cycle or an immersed circle which is a closed connected saturated subset of the union of separatrices and singular points. Recall that an immersed circle is the image of an immersion  $f : \mathbb{S}^1 \rightarrow S$ . A trivial circuit is a singular point. A periodic circuit is a periodic orbit. A circuit is either a trivial or non-trivial circuit. A non-trivial circuit  $\gamma$  is said to be limit if either  $\alpha(x) = \gamma$  or  $\omega(x) = \gamma$  for some point  $x \notin \gamma$ . A circuit  $\gamma$  is attracting (resp. repelling) if there is an open annulus  $\mathbb{A}$  such that a boundary component of  $\mathbb{A}$  is  $\gamma$  and that  $\mathbb{A} \subseteq W^s(\gamma)$  (resp.  $\mathbb{A} \subseteq W^u(\gamma)$ ). Then  $\mathbb{A}$  is called an attracting (resp. a repelling) collar basin of  $\gamma$ . Define the degree of a limit circuit  $\gamma$  with the collar basin  $\mathbb{A}$  the number of the separatrices connecting multi-saddles whose omega-limit set or alpha-limit set is  $\gamma$  and denote by  $\deg(\gamma)$  or  $\deg_{\mathbb{A}}(\gamma)$ . Note that a  $(\partial)$ -source is an attracting trivial circuit and a  $(\partial)$ -sink is a repelling trivial circuit. A non-periodic limit circuit  $\gamma$  is a strict limit circuit if either  $\gamma$  is one-sided or there are a collar basin  $\mathbb{A}$  of  $\gamma$  and a collar basin  $\mathbb{A}'$  of a limit circuit  $\mu$  with  $\mu \cap \gamma \neq \emptyset$  and  $\mathbb{A} \cap \mathbb{A}' = \emptyset$ . In other words, a non-periodic limit circuit  $\gamma$  is strict limit circuit if either  $\gamma$  is one-sided or there are separatrix  $\mu \subseteq \gamma$  and an open transverse arc  $T$  intersecting  $\mu$  such that  $f_v|_{\overline{T}_-}$  (resp.  $f_v|_{\overline{T}_+}$ ) is either contracting or repelling, where  $f_v : T \rightarrow T$  is the first return map,  $T_-$  and  $T_+$  are the two connected components of  $T \setminus \mu$ . A limit cycle is a limit circuit in  $\text{Per}(v)$ . A non-trivial circuit  $\gamma$  is a directed circuit if there are an oriented open annulus  $\mathbb{A}$  and a projection  $p : \mathbb{A} \rightarrow \mathbb{S}^1$  such that the circuit  $\gamma$  is a boundary component of  $\mathbb{A}$  and each fiber of  $p$  (i.e. the inverse image by  $p$  of a point) is an open transverse arc to  $v$ . Then the annulus  $\mathbb{A}$  is called an associated collar of  $\gamma$ . In other words, a circuit  $\gamma$  is directed if there is an open annulus  $\mathbb{A}$  such that a boundary component of  $\mathbb{A}$  is  $\gamma$  and the induced directed graph by  $\gamma$  (which is a simple closed curve) is a directed circle as a directed graph (i.e. the orientations

of edges are same). Note that each limit circuit is a directed circuit and that each collar basin is an associated collar.

**2.12. Flows of finite type.** A flow is of finite type if 1) it is quasi-regular, 2) there are at most finitely many limit cycles, and 3)  $LD \sqcup E = \emptyset$  (i.e. there are no quasi-minimal sets).

**2.13. Border points.** Denote by  $\partial_{\text{Per}}$  the union of periodic orbit in  $\partial S \cap \text{int Per}(v)$ . Denote by  $P_{\text{sep}}$  the union of multi-saddle separatrices and ss-separatrices in  $\text{int}P$ . Define  $P_{\text{lc}} := \bigcup\{\gamma \subset \text{int}P \sqcup \text{Sing}(v) : \text{limit circuit}\} \setminus \text{Sing}(v) \subseteq P_{\text{sep}}$ . In other words, the set  $P_{\text{lc}}$  is the intersection of  $\text{int}P$  and the union of limit circuits. Define the set  $\text{Bd}$  of border points by  $\text{Bd} := \partial \text{Sing}(v) \cup \partial \text{Per}(v) \cup \partial P \cup \partial LD \cup \partial E \cup P_{\text{sep}} \cup \partial_{\text{Per}}$ . Denote by  $\text{Per}_1$  the union of one-sided periodic orbits in  $\text{int}(\text{Per}(v) - \partial_{\text{Per}})$ . Define  $\text{BD} := \text{Bd} \sqcup \text{Per}_1$ .

**2.14. Orders.** A binary relation on a set is a pre-order if it is antisymmetric and transitive. A binary relation on a set is a partial order if it is reflexive, antisymmetric, and transitive. A poset is a set with a partial order.

**2.15. A cyclic order near a sink (resp. a source).** Define a cyclic relation  $\sim_c$  on a finite set  $F$  with  $n$  elements as follows:  $(O_1, O_2, \dots, O_n) \sim_c (O_{i_1}, O_{i_2}, \dots, O_{i_n})$  if  $F = \{O_1, O_2, \dots, O_n\}$  and there is an integer  $k = 0, 1, \dots, n-1$  such that  $j - i_j \equiv k \pmod n$ . An equivalent class is called a cyclic class of  $F$ .

Fix a  $n$ -saddle (resp. a sink or a source)  $x$  on a surface  $S$ . Let  $E_x$  be the set of separatrices of  $x$  (resp. between  $x$  and a multi-saddle and  $n := |E_x|$ ). Define a cyclic order  $<_c$  on  $E_x$  as follows: For an orientation  $\mu$  near  $x$ , a cyclic class  $[O_1, O_2, \dots, O_n]$  of  $E_x$  is counter-clockwise with respect to  $\mu$  if the separatrices  $O_1, \dots, O_{\deg \gamma}$  are arranged in the counter-clockwise direction around  $x$  with respect to  $\mu$ . Denote by  $[O_1, O_2, \dots, O_n]_\mu$  is a counter-clockwise cyclic class with respect to  $\mu$ . Define an equivalent relation  $\approx_c$  for counter-clockwise cyclic classes as follows:  $[O_1, O_2, \dots, O_n]_\mu \sim_c [O'_1, O'_2, \dots, O'_n]_{\mu'}$  if one of the following conditions holds

1)  $\mu = \mu'$  and  $[O_1, O_2, \dots, O_n]_\mu = [O'_1, O'_2, \dots, O'_n]_{\mu'}$  2)  $\mu$  and  $\mu'$  are opposite and  $[O_1, O_2, \dots, O_n]_\mu = [O'_n, O'_{n-1}, \dots, O'_1]_{\mu'}$ .

The equivalent class is called the cyclic class of  $x$  and denoted by  $[O_1, O_2, \dots, O_n]_x$ . Then the cyclic order  $<_c$  near  $x$  is defined by  $O_1 <_c O_2 <_c \dots <_c O_n <_c O_1$  (i.e.  $O_i <_c O_j$  if  $j - i \equiv 1 \pmod n$ ), where  $[O_1, O_2, \dots, O_n]_x$  is the cyclic class of  $x$ .

**2.16. A total order near a  $\partial$ -sink (resp. a  $\partial$ -source).** Fix a  $\partial$ - $n$ -saddle (resp. a  $\partial$ -sink or a  $\partial$ -source)  $x$  on a surface  $S$ . Let  $E_x$  be the set of separatrices of  $x$  (resp. between  $x$  and a multi-saddle and  $n := |E_x|$ ). Define a cyclic order  $<_t$  on  $E_x$  as follows: For an orientation  $\mu$  near  $x$ , a tuple  $(O_1, O_2, \dots, O_n)$  of  $E_x$  is counter-clockwise with respect to  $\mu$  if  $O_1 \sqcup O_n \subset \partial S$  and the separatrices  $O_1, \dots, O_{\deg \gamma}$  are arranged in the counter-clockwise direction around  $x$  with respect to  $\mu$ . Denote by  $[O_1, O_2, \dots, O_n]_\mu$  is a counter-clockwise cyclic class with respect to  $\mu$ . Define an equivalent relation  $\approx_c$  for counter-clockwise cyclic classes as follows:  $[O_1, O_2, \dots, O_n]_\mu \sim_c [O'_1, O'_2, \dots, O'_n]_{\mu'}$  if one of the following conditions holds

1)  $\mu = \mu'$  and  $[O_1, O_2, \dots, O_n]_\mu = [O'_1, O'_2, \dots, O'_n]_{\mu'}$   
2)  $\mu$  and  $\mu'$  are opposite and  $[O_1, O_2, \dots, O_n]_\mu = [O'_n, O'_{n-1}, \dots, O'_1]_{\mu'}$ .

The equivalent class is called the totally ordered class of  $x$  and denoted by  $[O_1, O_2, \dots, O_n]_x$ . Then the total order  $<_t$  near  $x$  is defined by  $O_1 <_t O_2 <_t \dots <_t O_n <_t O_1$  (i.e.

$O_i <_t O_j$  if  $j - i \equiv 1 \pmod{n}$ , where  $[O_1, O_2, \dots, O_n]_x$  is the totally ordered class of  $x$ .

**2.17. A cyclic order near a one-sided limit circuit.** Fix a one-sided limit circuit  $\gamma$  with the collar direction on a surface  $S$ . Let  $E_\gamma$  be the set of ss-separatrices each of whose  $\omega$ -limit or  $\alpha$ -limit set is  $\gamma$ . Put  $n := |E_\gamma|$ . Define a cyclic order  $<_c$  on  $E_\gamma$  as follows: A tuple  $(O_1, O_2, \dots, O_n)$  of  $E_\gamma$  is counter-clockwise with respect to  $\gamma$  if there is a closed transversal  $C$  near and parallel to  $\gamma$  and the intersections of  $U$  and the separatrices  $O_1, \dots, O_{\deg \gamma}$  are arranged in the flow direction of  $\gamma$  on  $C$ . Then the cyclic order  $<_c$  near  $\gamma$  is defined by  $O_1 <_c O_2 <_c \dots <_c O_n <_c O_1$  (i.e.  $O_i <_c O_j$  if  $j - i \equiv 1 \pmod{n}$ ), where  $(O_1, O_2, \dots, O_n)$  is counter-clockwise with respect to  $\gamma$ .

**2.18. A two-sided limit circuit with the collar direction.** Let  $\gamma$  be a two-sided limit circuit with a collar basin  $\mathbb{A}$  and  $L_\gamma$  the set of connected components of  $S - D_{ss}$  each of which intersects  $\mathbb{A}$ . Then the pair  $(\gamma, L_\gamma)$  is called a two-sided limit circuit with the collar direction.

**2.19. A cyclic order near a two-sided limit circuit.** Fix a two-sided limit circuit  $\gamma$  with the collar direction on a surface  $S$ . Let  $\mathbb{A}$  be an attracting (resp. repelling) collar basin of  $\gamma$  and  $E_\gamma$  the set of ss-separatrices which intersect  $\mathbb{A}$  and each of whose  $\omega$ -limit (resp.  $\alpha$ -limit) set is  $\gamma$ . Put  $n := |E_\gamma|$ . Define a cyclic order  $<_c$  on  $E_\gamma$  as follows: A tuple  $(O_1, O_2, \dots, O_n)$  of  $E_\gamma$  is counter-clockwise with respect to  $\gamma$  if there is a closed transversal  $C$  near and parallel to  $\gamma$  and the intersections of  $U$  and the separatrices  $O_1, \dots, O_{\deg \gamma}$  are arranged in the flow direction of  $\gamma$  on  $C$ . Then the cyclic order  $<_c$  near  $\gamma$  is defined by  $O_1 <_c O_2 <_c \dots <_c O_n <_c O_1$  (i.e.  $O_i <_c O_j$  if  $j - i \equiv 1 \pmod{n}$ ), where  $(O_1, O_2, \dots, O_n)$  is counter-clockwise with respect to  $\gamma$ .

**2.20. Heights of orbits.** For a subset  $A$  and a point  $x$  of a topological space  $(X, \tau)$ , an abbreviated form of the singleton  $\{x\}$  (resp. the difference  $A - \{x\}$ , the point closure  $\overline{\{x\}}$ ) will be  $x$  (resp.  $A - x, \overline{x}$ ). The specialization pre-order  $\leq_\tau$  on a topological space  $(X, \tau)$  is defined as follows:  $x \leq_\tau y$  if  $x \in \overline{y}$ . For a point  $x$  of  $(X, \tau)$ , define the upset  $\uparrow x := \{y \in P \mid x \leq_\tau y\}$ , the downset  $\downarrow x := \{y \in P \mid y \leq_\tau x\}$ , and the class  $\hat{x} := \downarrow x \cap \uparrow x$ . Then the set of classes is a decomposition and so its quotient space is a  $T_0$  space, which is called the  $T_0$ -identification of  $X$  and denoted by  $\hat{X}$ . Denote by  $\max X$  (resp.  $\min X$ ) the set of maximal (resp. minimal) points. Note that  $\downarrow x = \overline{x}$  and that the set  $\min X$  of minimal elements in  $X$  is the set of points whose classes are closed. Define the upset  $\uparrow A := \bigcup_{x \in A} \uparrow x$ , the downset  $\downarrow A := \bigcup_{x \in A} \downarrow x$ , the class  $\hat{A} := \bigcup_{x \in A} \hat{x}$ ,  $A^\uparrow = \uparrow A - \hat{A}$ , and  $A^\downarrow = \downarrow A - \hat{A}$ . Notice that  $x^\uparrow = \uparrow x - \hat{x}$ ,  $x^\downarrow = \downarrow x - \hat{x}$ ,  $\downarrow A \subseteq \overline{A}$ , and  $A^\downarrow \subseteq \overline{A} - A = \sigma A$ . Define the height  $\text{ht}_\tau x$  of  $x$  by  $\text{ht}_\tau x := \sup\{\#C - 1 \mid C : \text{chain containing } x \text{ as the maximal point}\}$ . Recall that a chain is a totally ordered subset of a pre-ordered set. The height  $\text{ht}_\tau A$  of a subset  $A \subseteq X$  is defined by  $\text{ht}_\tau A := \sup_{x \in A} \text{ht}_\tau x$ .

**2.21. Orbit (class) spaces.** For a flow  $v$  on a compact surface  $S$ , denote by  $(S/v, \tau_v)$  (resp.  $(S/\hat{v}, \tau_{\hat{v}})$ ) the orbit space (resp. the orbit class space). Note that the orbit class space  $S/\hat{v}$  is the  $T_0$ -identification of the orbit space  $S/v$ . For a point  $x \in S$ , define the height of  $x$  by the height of the point  $\hat{x}$  in  $S/\hat{v}$  and write

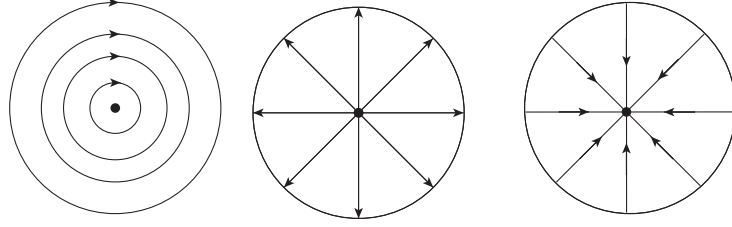
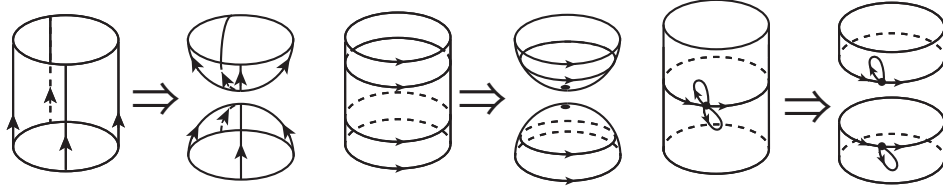


FIGURE 1. A center disk, a sink disk, and a source disk


 FIGURE 2. An operation  $C_t$ , an operation  $C_o$ , and an operation  $C_d$ 

$\text{ht}(x) := \text{ht}_{\tau_{\hat{v}}}(\hat{x})$ . Then the compatible pre-order  $\leq$ , called the specialization pre-order of the quotient topology  $\tau_{\hat{v}}$ , on  $S$  is defined as follows:  $x \leq y$  if  $\hat{x} \leq_{\tau_{\hat{v}}} \hat{y}$ . Denote by  $S_k$  the set of points of height  $k$ . The height  $\text{ht}_{\tau} A$  of a subset  $A \subseteq S$  is defined by  $\text{ht} A := \sup_{x \in A} \text{ht}(x)$ . The orbit space  $T/v$  (resp. orbit class space  $T/\hat{v}$ ) of a saturated subset  $T$  of  $S$  is a quotient space  $T/\sim$  defined by  $x \sim y$  if  $O(x) = O(y)$  (resp.  $\overline{O(x)} = \overline{O(y)}$ ).

**2.22. Center disks, sink disks, source disks.** A closed disk is a center disk if it is a union of one center and periodic orbits. A closed disk is a sink (resp. source) disk if the boundary of it is a closed transverse and the interior of it consists of one sink (resp. source) and of non-closed proper orbits (see Figure 1).

**2.23. Operations.** Define an operation  $C_t$  as cutting an essential closed transversal and pasting one sink disk and one source disk, an operation  $C_o$  as cutting an essential periodic orbit and pasting one or two center disks, and an operation  $C_d$  as removing an essential one-sided (resp. two-sided) loop in  $D$  and pasting a double covering (resp. two copies) of the loop to the new boundary (resp. two boundaries) (see Figure 2). Recall that a Cherry blow-up operation replaces a flow box with a Cherry flow box as Figure 3. Denote by  $\text{Ch}_l$  an Cherry blow-up operation to a limit cycle (i.e. replacing a limit cycle by a Cherry flow with a homoclinic separatrix of the saddle) as Figure 4. We can define an Cherry blow-up operation to a directed circuit as Figure 4 and also denote by  $\text{Ch}_l$ .

**2.24. Graphs.** Recall an ordered triple  $G := (V, E, r)$  is an abstract multi-graph if  $V$  and  $E$  are sets and  $r : E \rightarrow \{\{x, y\} \mid x, y \in V\}$ . A graph is a cell complex whose dimension is at most one and which is a geometric realization of an abstract multi-graph. Note that a finite (directed) multi-graph can be embedded in a surface. In other words, it can be drawn in such a way that no edges cross each other. Such a drawing is called a surface (directed) graph. Moreover a disjoint union of a surface

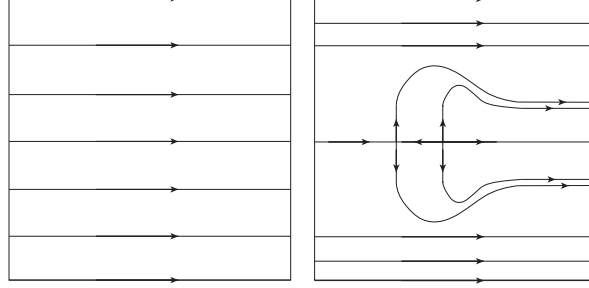


FIGURE 3. A flow box and a Cherry flow box

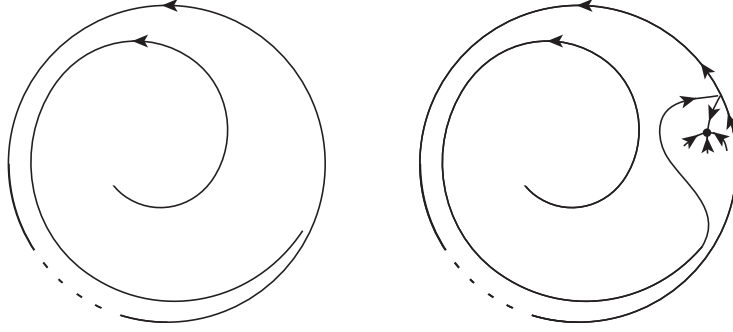


FIGURE 4. An Cherry blow-up operation to an attracting directed circuit

directed graph and finite simple closed directed curves embedded in a surface is called a generalized surface directed graph.

**2.25. Reeb domains.** An open saturated subset  $U$  of a surface  $S$  is called a Reeb domain with respect to a flow on  $S$  if there are two limit circuits  $\gamma_-$ ,  $\gamma_+$ , and an open annulus  $\mathbb{A}$  with  $\partial\mathbb{A} = \gamma_- \sqcup \gamma_+$  and  $U \subseteq \mathbb{A} \cap W^u(\gamma_-) \cap W^s(\gamma_+)$  such that the flow directions of  $\gamma_-$  and  $\gamma_+$  with respect to the metric completion of  $\mathbb{A}$  are opposite, where  $W^u(\gamma_-)$  is the unstable manifold of  $\gamma_-$  and  $W^s(\gamma_+)$  is the stable manifold of  $\gamma_+$ .

### 3. PROPERTIES OF SURFACE FLOWS

Let  $v$  be a flow on a compact connected surface  $S$ . We characterize shells.

**Lemma 3.1.** *The following statements hold for a flow  $v$  on a compact surface  $S$ :*

- 1)  $\overline{\text{LD}} - \text{LD} \subseteq \delta \text{Sing}(v) \sqcup \delta \text{P}.$
- 2)  $\overline{\text{E}} - \text{E} \subseteq \delta \text{Sing}(v) \sqcup \delta \text{P}.$
- 3)  $\overline{\text{Per}(v)} - \text{Per}(v) \subseteq \delta \text{Sing}(v) \sqcup \delta \text{P}.$

*In other words, we have  $\sigma \text{Per}(v) \sqcup \sigma \text{LD} \sqcup \sigma \text{E} \subseteq \delta \text{Sing}(v) \sqcup \delta \text{P}.$*

*Proof.* It's known that the total number of quasi-minimal sets for  $v$  is bounded [6, 8]. By Proposition 2.2 [17], the assertions 1) and 2) hold. Lemma 2.3 [17] implies that  $\overline{\text{Per}(v)} \cap (\text{LD} \sqcup \text{E}) = \emptyset$  and so  $\overline{\text{Per}(v)} \subseteq \delta \text{Sing}(v) \sqcup \text{Per}(v) \sqcup \delta \text{P}.$   $\square$



**Lemma 3.2.**  $\delta P = (\sigma \text{Per}(v) \sqcup \sigma \text{LD} \sqcup \sigma E) \setminus \text{Sing}(v)$ .

*Proof.* Lemma 3.1 implies that  $(\sigma \text{Per}(v) \sqcup \sigma \text{LD} \sqcup \sigma E) \setminus \text{Sing}(v) \subseteq \delta P$ . Conversely, the closedness of  $\text{Sing}(v)$  implies that  $\delta P \subseteq \sigma(S - P) = \sigma(\text{Sing}(v) \sqcup \text{Per}(v) \sqcup \text{LD} \sqcup E) = \sigma(\text{Per}(v) \sqcup \text{LD} \sqcup E) \setminus \text{Sing}(v) \subseteq (\sigma \text{Per}(v) \sqcup \sigma \text{LD} \sqcup \sigma E) \setminus \text{Sing}(v)$ .  $\square$

**Lemma 3.3.** *For any point  $x \in \text{LD} \sqcup E$ , there is a closed transverse through  $x$ . Moreover each closed transversal through a point in  $\text{LD} \sqcup E$  is essential.*

*Proof.* Fix a point  $x \in \text{LD} \sqcup E$  and a transverse arc  $I$  such that  $x$  is the interior point of  $I$ . Fix a path  $C$  in the orbit  $O(x)$  whose boundary is contained in  $I$ . By the water fall construction, there is a closed transversal  $\gamma$  intersecting  $x$  near the union of the path  $C$  and the transverse sub-arc  $J$  of  $I$  with  $\partial C = \partial J$ . Suppose that  $\gamma$  is not essential. Let  $S^*$  be the resulting closed surface from  $S$  by collapsing all boundaries into singletons. Then  $\gamma$  is null homotopic in  $S^*$  and so the point  $x \in P$ , which contradicts to the weak recurrence of  $x$ . a closed annulus.  $\square$

We obtain the following description of a neighborhood of  $\text{Per}(v)$ .

**Lemma 3.4.** *The union  $\text{Per}(v) \sqcup \text{int}P$  is an open neighborhood of  $\text{Per}(v)$ . In particular,  $\text{Per}(v) \cap \overline{\delta P} = \emptyset$ .*

*Proof.* Taking the orientation double covering (resp. the double of a manifold), we may assume that  $S$  is closed orientable. Fix a point  $x \in \text{Per}(v)$ . Take a collar  $U \subseteq S - \text{Sing}(v)$  of  $O(x)$ . The flow box theorem implies that there are an open annulus  $\mathbb{A} \subseteq U$  with  $O(x) \subseteq \partial \mathbb{A}$  and a transverse closed arc  $T \subseteq \overline{\mathbb{A}}$  such that  $x$  is contained in  $\partial T$ . Identify  $T$  with  $[0, 1]$  (resp.  $x$  with 0). Let  $f_v : T \rightarrow T$  be the first return map induced by  $v$ . Then  $f_v(0) = 0$ . If  $f_v : T \rightarrow T$  is contracting or expanding near 0 on each side, then  $O$  is a limit cycle and so  $x \in \text{int}(\text{Per}(v) \sqcup \text{int}P)$ . Thus we may assume that  $f_v : T \rightarrow T$  is neither contracting nor expanding near 0 on each side. This implies that 0 is an accumulation point of the fixed point set  $\text{Fix}(f_v)$ . Shortening  $T$ , we may assume that  $f_v(1) = 1$ . Then the saturation of  $T$  is a closed annulus. Then each orbit intersecting  $\text{Fix}(f_v)$  is contained in  $\text{Per}(v)$  and each orbit intersecting  $T - \text{Fix}(f_v)$  is contained in  $\text{int}P$ . This implies that  $\text{Per}(v) \sqcup \text{int}P$  is a neighborhood of  $x$  and so of  $\text{Per}(v)$ .  $\square$

**Lemma 3.5.**  $\delta \text{Per}(v) \subseteq \overline{\text{int}P}$  is contained in the closure of the union of limit cycles.

*Proof.* Obviously, each limit cycle is contained in  $\delta \text{Per}(v)$ . Taking the orientation double covering, we may assume that  $S$  is orientable. By Lemma 3.4, the union  $\text{Per}(v) \sqcup \text{int}P$  is an open neighborhood of  $\text{Per}(v)$ . Then  $\delta \text{Per}(v) \subset \overline{\text{int}P}$ . Fix a periodic orbit  $O \subseteq \delta \text{Per}(v)$ . If  $O$  is a limit cycle, then  $O \subseteq \overline{\text{int}P}$ . Thus we may assume that  $O$  is not a limit cycle. Since  $O \cap \text{int} \text{Per}(v) = \emptyset$ , there is a collar  $U$  of it which is contained in  $\text{Per}(v) \sqcup \text{int}P$ . The flow box theorem implies that there are an open annulus  $\mathbb{A} \subseteq U$  with  $O \subseteq \partial \mathbb{A}$  and a closed transverse arc  $T \subseteq \overline{\mathbb{A}}$  such that  $O \cap T$  is contained in  $\partial T$ . Identify  $T$  with  $[0, 1]$  (resp.  $O \cap T$  with 0). Let  $f_v : T \rightarrow T$  be the first return map induced by  $v$ . Then  $f_v(0) = 0$ . Since  $O$  is not a limit cycle,  $f_v : T \rightarrow T$  is neither contracting nor repelling near 0 on each side. This implies that 0 is an accumulation point of the fixed point set  $\text{Fix}(f_v)$ . Since  $O \subseteq \overline{\text{int}P}$ , the transverse  $T$  has a convergence sequence to 0 of fixed points which are either attracting or repelling from one side. This means that  $O$  is contained in the closure of the union of limit cycles.  $\square$

We obtain the following description of a neighborhood of  $E$  (resp.  $LD$ ).

**Lemma 3.6.**  $E \subseteq \text{int}(P \sqcup E)$ .

*Proof.* Lemma 3.1 implies  $E \cap \overline{\text{Per}(v) \sqcup LD} = \emptyset$ . The closedness of  $\text{Sing}(v)$  implies  $E \subseteq S - \overline{\text{Sing}(v) \sqcup \text{Per}(v) \sqcup LD} = \text{int}(P \sqcup E)$ .  $\square$

**Lemma 3.7.**  $LD \subseteq \text{int}(P \sqcup LD)$ .

*Proof.* Lemma 3.1 implies  $LD \cap \overline{\text{Per}(v) \sqcup E} = \emptyset$ . The closedness of  $\text{Sing}(v)$  implies  $LD \subseteq S - \overline{\text{Sing}(v) \sqcup \text{Per}(v) \sqcup E} = \text{int}(P \sqcup LD)$ .  $\square$

Note that  $\text{int}P \sqcup E$  is not a neighborhood of  $E$  and that  $\text{int}P \sqcup LD$  is not a neighborhood of  $LD$  in general. For instance, there is a toral flow such that  $\text{int}P \sqcup E$  is not a neighborhood of  $E$ . In fact, consider a Denjoy flow  $w$  on a torus with an exceptional minimal set  $\mathcal{M}$ . Fix an orbit  $O \subset \mathcal{M}$ . Choose a point  $x \in O$  and a sequence  $(t_n)_{n \in \mathbb{Z}}$  on  $O$  with  $\lim_{n \rightarrow \infty} t_n = \infty$ ,  $\lim_{n \rightarrow -\infty} t_n = -\infty$ , and  $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow -\infty} x_n = x$ , where  $x_n := w_{t_n}(x)$ . Using the bump function  $\varphi$  with  $\varphi^{-1}(0) = \{x_n \mid n \in \mathbb{Z}\}$ , replace  $O$  with a union of singular points and separatrices of the resulting flow  $v_\varphi$  (i.e.  $O = \text{Sing}(v_\varphi) \sqcup \{\text{separatrix of } v_\varphi\}$ ) such that  $O_{v_\varphi}(y) = O_v(y)$  for any point  $y \in \mathbb{T}^2 - O$ . Then  $\mathbb{T}^2 = \text{Sing}(v_\varphi) \sqcup P(v_\varphi) \sqcup E(v_\varphi)$  where  $P(v_\varphi)$  is the union of non-closed proper orbits of  $v_\varphi$  and  $E(v_\varphi)$  is the union of exceptional orbits of  $v_\varphi$ . Since  $\mathcal{M} = \overline{O} = \overline{E(v_\varphi)}$ , each neighborhood of  $O$  contains  $E(v_\varphi)$  and each neighborhood of  $E(v_\varphi)$  contains  $O$ . Since  $O \subset \text{Sing}(v_\varphi) \sqcup P(v_\varphi)$ , we obtain  $O \cap (\text{int}P(v_\varphi) \sqcup E(v_\varphi)) = \emptyset$ . Since each neighborhood of  $E(v_\varphi)$  contains  $O$ , we have  $\text{int}P(v_\varphi \sqcup E(v_\varphi))$  is not a neighborhood of  $E(v_\varphi)$ . Moreover, there is a toral flow  $w$  with countable singular points and with  $\overline{\text{Cl}(v)} \neq \Omega(v)$  such that  $\text{int}P \sqcup LD$  is not a neighborhood of  $LD$ . Indeed, consider an irrational toral flow  $w$ . Fix an orbit  $O$  and a point  $x \in O$ . Choose a point  $x \in O$  and a sequence  $(t_n)_{n \in \mathbb{Z}}$  on  $O$  with  $\lim_{n \rightarrow \infty} t_n = \infty$ ,  $\lim_{n \rightarrow -\infty} t_n = -\infty$ , and  $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow -\infty} x_n = x$ , where  $x_n := w_{t_n}(x)$ . Replacing  $O$  by a union of singular points and separatrices, we obtain the resulting flow  $v$  on the torus  $\mathbb{T}^2$  such that the singular point set  $\text{Sing}(v) = \{x\} \sqcup \{x_n\}_{n \in \mathbb{Z}}$  is countable,  $S - O = LD$ ,  $O = \text{Sing}(v) \sqcup P$ , and  $\Omega(v) = \mathbb{T}^2 \neq \text{Sing}(v) = \overline{\text{Cl}(v)}$ . Since  $\overline{\mathbb{T}^2 - LD} = \overline{\text{Sing}(v) \sqcup P} = \overline{O} = \mathbb{T}^2$  and  $LD \neq \emptyset$ , the union  $LD$  is not open. Summaries descriptions of neighborhoods as follows.

**Theorem 3.8.** *Let  $v$  be a flow on a compact surface  $S$ . The following statements hold:*

1.  $\text{Per}(v) \subseteq \text{int}(\text{Per}(v) \sqcup \text{int}P) = \text{Per}(v) \sqcup \text{int}P$ .
2.  $E \subseteq \text{int}(P \sqcup E)$ .
3.  $LD \subseteq \text{int}(P \sqcup LD)$ .

#### 4. HEIGHTS OF ORBITS

Let  $v$  be a flow on a compact connected surface  $S$ . We characterize superior elements.

**Lemma 4.1.** *The following statements hold:*

- 1)  $\text{Sing}(v)^\uparrow \subseteq P \sqcup LD \sqcup E$ .
- 2)  $\text{Per}(v)^\uparrow \subseteq \text{int}P$ .
- 3)  $P^\uparrow \subseteq LD \sqcup E$ .
- 4)  $LD^\uparrow = \emptyset$ .
- 5)  $E^\uparrow \subseteq \text{int}P$ .

*Proof.* Since closed orbits are minimal sets, the assertions 1) and 3) hold. Lemma 2.1 [17] implies  $\text{Per}(v)^\uparrow \subseteq \text{int}P$ . For an orbit  $O$  with  $\overline{O} \cap \text{LD} \neq \emptyset$ , there is an orbit  $O' \subset \text{LD} \cap \overline{O}$ . Since the closure  $\overline{O'}$  is a neighborhood of  $O'$ , we have  $O \cap \overline{O'} \neq \emptyset$  and so  $\overline{O} = \overline{O'}$ . This means that  $O \subset \text{LD}$  and so  $\text{LD}^\uparrow = \emptyset$ . Lemma 2.3 [17] implies that  $E \subset \text{int}(P \sqcup E)$ . Since  $\text{int}P \cap \overline{E} = \emptyset$ , we have  $E^\uparrow = (\uparrow E) \setminus \overline{E} \subseteq \text{int}(P \sqcup E) \setminus \overline{E} \subseteq \text{int}P$ .  $\square$

**Corollary 4.2.** *The following statements hold:*

- 1)  $x^\uparrow \subseteq P \sqcup \text{LD} \sqcup E$  for any  $x \in \text{Sing}(v)$ .
- 2)  $x^\uparrow \subseteq \text{int}P$  for any  $x \in \text{Per}(v)$ .
- 3)  $x^\uparrow \subseteq P \sqcup \text{LD} \sqcup E$  for any  $x \in P$ .
- 4)  $x^\uparrow = \emptyset$  for any  $x \in \text{LD}$ .
- 5)  $\emptyset \neq x^\uparrow \subseteq \text{int}P$  for any  $x \in E$ .

*Proof.* Since the orbit class of a closed orbit is the orbit, the assertions 1) and 2) hold. The previous lemma implies 3) obviously. Proposition 2.2 [17] implies the assertions 4) and 5).  $\square$

We characterize inferior elements.

**Lemma 4.3.** *The following statements hold:*

- 1)  $\text{Sing}(v)^\downarrow = \emptyset$  (i.e.  $\text{Sing}(v) \subseteq \min S$ ).
- 2)  $\text{Per}(v)^\downarrow = \emptyset$  (i.e.  $\text{Per}(v) \subseteq \min S$ ).
- 3)  $P^\downarrow \subseteq S - \text{LD}$  and  $(P \setminus (\text{Per}(v) \sqcup E)^\uparrow)^\downarrow \subseteq \text{Sing}(v) \sqcup P$ .
- 4)  $\text{LD}^\downarrow \subseteq \text{Sing}(v) \sqcup \delta P$ .
- 5)  $E^\downarrow \subseteq \text{Sing}(v) \sqcup \delta P$ .

*Proof.* Since closed orbits are minimal sets, the assertions 1) and 2) hold. Lemma 4.1 implies the rest statements.  $\square$

**Corollary 4.4.** *The following statements hold:*

- 1)  $x^\downarrow = \emptyset$  for any  $x \in \text{Cl}(v)$ .
- 2)  $\emptyset \neq x^\downarrow \subseteq S - \text{LD}$  for any  $x \in P$ .
- 3)  $\emptyset \neq x^\downarrow \subseteq \text{Sing}(v) \sqcup \delta P$  for any  $x \in \text{LD} \sqcup E$ .

*Proof.* The previous lemma implies the assertion 1). Proposition 2.2 [17] implies the assertions 2) and 3).  $\square$

**Lemma 4.5.** *The following statements hold:*

- 1) The set  $\min S$  of height zero points is the union of minimal sets.
- 2)  $\text{Cl}(v) \subseteq \min S \subseteq S - P$ .
- 3)  $\text{int Cl}(v) \sqcup (\text{int}P - P_{\text{lc}}) \sqcup \text{LD} \subseteq \max S$ .
- 4)  $E \cap \max S = \emptyset$  if  $|\text{Sing}(v)| < \infty$ .

*Proof.* The definition of height implies 1). Since closed orbits are minimal sets, the definition of  $P$  implies the assertion 2). Lemma 4.1 and the definition of a limit circuit imply 3). By Proposition 2.1 [9] (cf. Lemma 3 [15]), the finiteness of singular points implies 4).  $\square$

## 5. FINITENESS OF SINGULAR POINTS

Let  $v$  be a flow on a compact surface  $S$ . In this section, we assume that the flow  $v$  has finitely many singular points.

**5.1. Properties of orbits.** By dimension, we mean the small inductive dimension. By Urysohn's theorem, the Lebesgue covering dimension, the small inductive dimension, and the large inductive dimension are corresponded in normal spaces. A compact metrizable space  $X$  whose inductive dimension is  $n > 0$  is an  $n$ -dimensional Cantor-manifold if the complement  $X - L$  for any closed subset  $L$  of  $X$  whose inductive dimension is less than  $n - 1$  is connected. It's known that a compact connected manifold is a Cantor-manifold [4, 16].

**Lemma 5.1.** *Each point  $x \in \text{Sing}(v) \cap \overline{\text{Per}(v)}$  is either a compressed center or contained in a directed circuit.*

*Proof.* Suppose that a point  $x \in \text{Sing}(v) \cap \overline{\text{Per}(v)}$  is not a compressed center. Taking the orientation double covering (resp. the double of a manifold), we may assume that  $S$  is closed orientable. Cutting an essential periodic orbit and pasting two center disks, we may assume that each periodic orbit is null homotopic. The finiteness of  $\text{Sing}(v)$  implies that the singular point  $x$  is isolated. Since  $x$  is not a compressed center, there is a small open neighborhood  $U$  of  $x$  which contains no periodic orbits such that  $U \cap \text{Sing}(v) = \{x\}$ . By Lemma 3.3, the neighborhood  $U$  contains no quasi-minimal sets. Since  $x \in \text{Per}(v)$ , there are a sequence  $(O_n)_{n>0}$  of periodic orbits and a sequence  $(x_n)$  with  $x \in O_n$  converging to  $x$ . Then  $U \cap O_n \neq \emptyset$  for any  $n > 0$ . Since each periodic orbit is null homotopic, it bounds a singular point. Since  $|\text{Sing}(v)| < \infty$ , taking a sub-sequence of  $(O_n)_{n>0}$ , we may assume that there are open annuli  $\mathbb{A}_n$  ( $n > 0$ ) whose boundary is the union  $O_n \sqcup O_{n+1}$  such that  $\mathbb{A}_n \cap \mathbb{A}_m = \emptyset$  for any  $n \neq m$ . Then  $\mathbb{A} := \bigcup_{n>0} (O_n \sqcup \mathbb{A}_n) = \bigcup_{n>0} \overline{\mathbb{A}_n}$  is an open annulus with  $x \in \partial$ , where  $\partial$  is a boundary component of  $\mathbb{A}$ . Since  $\partial$  is invariant and  $x \in \partial \cap U$ , we obtain  $\partial \not\subseteq U$ . Because the dimension of  $\mathbb{A}$  is two, the dimension of the boundary component  $\partial$  is at most one. Then both  $U \cap \mathbb{A}$  and  $U \setminus \overline{\mathbb{A}}$  are non-empty open and so  $U - (U \cap \partial) = U \setminus \partial$  is disconnected. Since  $S$  is a Cantor-manifold, the dimension of the boundary component  $\partial$  is one. Lemma 3.1 implies that  $\partial \subset \overline{\text{Per}(v)} \subseteq \text{Sing}(v) \sqcup \text{Per}(v) \sqcup \delta P$ . If there is a point  $y \in \partial \cap \text{Per}(v)$ , then the flow box theorem implies that the periodic orbit  $O(y) \subset \partial$  is parallel to  $O_n$  for any  $n > 0$  and so  $O(y) = \partial$ , which contradicts to  $x \in \partial \cap \text{Sing}(v)$ . Thus  $\partial \subseteq \text{Sing}(v) \sqcup \delta P$ . Since the orbit closure of a point in  $\partial \cap \delta P$  is contained in  $\partial$ , the boundary component  $\partial$  is a finite union of connecting separatrices and singular points.  $\square$

The finiteness of singular points is necessary. In other words, there is a point  $x \in \text{Sing}(v) \cap \overline{\text{Per}(v)}$  is neither a compressed center nor contained in a directed circuit. Indeed, consider a point  $x_0 \in \text{int}P$  of a flow  $v_0$  and a sequence  $(x_{1/n})_{n \in \mathbb{Z}_{>0}}$  (resp.  $(x_{-1/n})_{n \in \mathbb{Z}_{>0}}$ ) in  $O(x_0)$  converging from the positive (resp. negative) side in  $O(x_0)$ . Replacing an orbit  $O(x_0)$  with singular points  $\{x_0\} \sqcup \{x_{-1/n}, x_{1/n} \mid n \in \mathbb{Z}_{>0}\}$  and non-closed proper orbits and blow up the singular points  $x_{-1/n}$  (resp.  $x_{1/n}$ ) (i.e. replacing  $x_{-1/n}$  (resp.  $x_{1/n}$ ) by a center disk whose boundary is a union of a homoclinic separatrix and a saddle (see Fig. 5)) such that the diameters of center disks converge to zero if  $|n|$  goes to infinity, the resulting flow  $v$  have a point  $x \in \text{Sing}(v) \cap \overline{\text{Per}(v)}$  is neither a compressed center nor contained in a directed circuit.

In general, the union LD of a flow is not open even if singular points are countable. In fact, blow up (i.e. replacing isolated singular points with center disks) of a toral flow with  $\partial \text{LD} = \partial P = \mathbb{T}^2$  (cf. Example 2.10 [17]) such that isolated

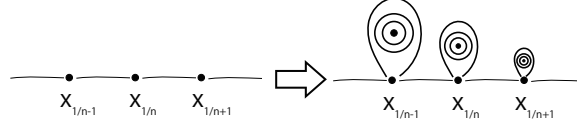


FIGURE 5. Blow up operations for 0-saddles

singular points converge to a singular point, the resulting flow is desired. On the other hand, the finiteness implies the following statement.

**Lemma 5.2.** *We have  $\text{LD} \cap \overline{\text{P}} = \emptyset$ .*

*Proof.* Assume that there is a point  $x \in \text{LD} \cap \overline{\text{P}}$ . Since the orbit closure  $\overline{O(x)}$  is a neighborhood of  $x$ , there is a sequence  $(x_n)_{n \in \mathbb{Z}_{>0}}$  in  $\overline{O(x)} \cap \text{P} \subseteq \overline{\text{LD}}$  converging to  $x$  such that  $O(x_n) \cap O(x_m) = \emptyset$  for any  $n \neq m \in \mathbb{Z}_{>0}$ . We show that  $\overline{O(x_n)} - O(x_n)$  consists of one or two singular points. Indeed, by a generalization of the Poincaré-Bendixson theorem (Theorem 2.6.1 [11]), the  $\omega$ -limit set of  $O(x_n)$  is either a singular point, an attracting limit circuit, or a quasi-minimal set. Assume that  $\omega(x_n)$  is an attracting limit circuit. Since each orbit whose closure intersects an attracting limit circuit is proper, the inclusion  $O(x_n) \subseteq \overline{O(x)}$  implies that the locally dense orbit  $O(x)$  is proper, which contradicts. Assume that  $\omega(x_n)$  is a quasi-minimal set. Since  $O(x_n)$  is proper, the  $\omega$ -limit set  $\omega(x_n)$  is not  $\overline{O(x)}$ . Since  $\omega(x_n) \subseteq \overline{O(x)}$ , Proposition 2.2 [17] implies that  $\omega(x_n) = \overline{O(x)}$ , which contradicts. Thus  $\omega(x_n)$  is a singular point. By symmetry, the  $\alpha$ -limit set  $\alpha(x_n)$  is a singular point. Since  $|\text{Sing}(v)| < \infty$ , there are singular points  $\alpha, \omega$  and a subsequence  $(x_{k_n})_{n \in \mathbb{Z}_{>0}}$  of  $(x_n)_{n \in \mathbb{Z}_{>0}}$  such that  $\omega(x_{k_n}) = \omega$  and  $\alpha(x_{k_n}) = \alpha$ . Since  $S$  is compact, there are an open disk  $D \subseteq \overline{O(x)}$  and three points  $x_{n_1}, x_{n_2}, x_{n_3}$  in the subsequence such that  $\partial D = \overline{O(x_{n_1})} \cup \overline{O(x_{n_3})}$  and  $x_{n_2} \in \text{int} D$ . Then  $\overline{\text{LD}} \cap \text{int} D = \emptyset$  and so  $x_{n_2} \notin \overline{\text{LD}}$ , which contradicts to the choice of  $x_{n_2}$ . Thus  $\text{LD} \cap \overline{\text{P}} = \emptyset$ .  $\square$

We show the openness of LD.

**Lemma 5.3.** *The union LD is open.*

*Proof.* Since the singular point set  $\text{Sing}(v)$  is closed, Lemma 3.1 and Lemma 5.2 imply that  $\overline{\text{Sing}(v)} \sqcup \overline{\text{Per}(v)} \sqcup \text{P} \sqcup \text{E} \cap \text{LD} = \emptyset$  and so that the union LD is open.  $\square$

The finiteness of singular points implies the existence of collar basins of limit circuits.

**Lemma 5.4.** *For any limit circuit, there is a collar basin of it.*

*Proof.* Fix a limit circuit  $\gamma$ . Since  $|\text{Sing}(v)| < \infty$ , there is an open neighborhood  $U$  of  $\gamma$  such that  $(U - \gamma) \cap \text{Sing}(v) = \emptyset$ . Since a limit circuit is directed, there are an associated collar  $\mathbb{A} \subset U$  of  $\gamma$  and a point  $x \in \mathbb{A}$  whose orbit closure contains  $\gamma$ . By the time reversion if necessary, we may assume that  $\omega(x) = \gamma$ . The flow box theorem implies that  $\omega(y) = \gamma$  for any point  $y \in \mathbb{A}$ . This means that  $\mathbb{A}$  is a collar basin of  $\gamma$ .  $\square$

Note that each collar basin does not intersect  $\overline{\text{LD} \sqcup \text{E}}$ .

**Lemma 5.5.** *An orbit contained in  $\text{P} \cap \overline{\text{LD} \sqcup \text{E}}$  is a connecting separatrix.*

*Proof.* Fix a point  $x \in P \cap \overline{LD \sqcup E}$  and an orbit  $O \subseteq LD \sqcup E$  with  $x \in \overline{O}$ . By a generalization of the Poincaré-Bendixson theorem, the  $\omega$ -limit set of  $x$  is either a singular point, an attracting limit circuit, or a quasi-minimal set. Since each orbit whose closure intersects an attracting limit circuit is proper, the  $\omega$ -limit set  $\omega(x)$  is not an attracting limit circuit. Assume that  $\omega(x)$  is a quasi-minimal set. Since  $x$  is proper, the  $\omega$ -limit set  $\omega(x)$  does not contain  $x$  and so  $\omega(x) \neq \overline{O}$ . Since  $\omega(x) \subseteq \overline{O}$ , Proposition 2.2 [17] implies that  $\omega(x) = \overline{O}$ , which contradicts. Thus the  $\omega$ -limit set  $\omega(x)$  is a singular point. By the symmetry, the  $\alpha$ -limit set  $\alpha(x)$  is a singular point.  $\square$

We describe the border  $\delta P$  as follows.

**Proposition 5.6.** *Each orbit in  $\delta P$  is a connecting separatrix. In particular, we have  $\text{ht}(\delta P) = 1$  if  $\delta P \neq \emptyset$ .*

*Proof.* Fix an orbit  $O$  in  $\delta P$ . Since  $\delta P = P - \text{int}P = P \cap \overline{\text{Sing}(v) \sqcup \text{Per}(v) \sqcup LD \sqcup E} = P \cap \overline{\text{Per}(v) \sqcup LD \sqcup E}$ , we have  $O \subseteq P \cap \overline{\text{Per}(v) \sqcup LD \sqcup E}$ . By Lemma 5.5, we may assume that  $O \subseteq P \cap \overline{\text{Per}(v)}$ . Then  $\overline{O} \cap (LD \sqcup E) = \emptyset$ . By a generalization of the Poincaré-Bendixson theorem, the  $\omega$ -limit (resp.  $\alpha$ -limit) set of  $O$  is either a singular point, or a limit circuit. Since  $O \subseteq \delta P$ , the  $\omega$ -limit (resp.  $\alpha$ -limit) set of  $O$  contains no limit circuits and so it is a singular point. Therefore  $O$  is a connecting separatrix.  $\square$

**Corollary 5.7.**  $E \subset \text{int}(E \sqcup \text{int}P)$ .

*Proof.* By Lemma 5.6, the border  $\delta P$  consists of finitely many connecting separatrices. Theorem 3.8 implies that  $\text{int}(E \sqcup P)$  is an open neighborhood of  $E$ . Then so is the difference  $\text{int}(E \sqcup P) \setminus \delta P = \text{int}(E \sqcup \text{int}P)$ .  $\square$

**Lemma 5.8.** *The difference  $\overline{LD} - LD$  (resp.  $\overline{E} - E$ ) is a finite unions of connecting separatrices and singular points. In particular, we have  $\text{ht}(LD \sqcup E) \leq 2$ .*

*Proof.* Lemma 3.1 implies  $\overline{LD} - LD \subseteq \text{Sing}(v) \sqcup \delta P$  and  $\overline{E} - E \subseteq \text{Sing}(v) \sqcup \delta P$ . By the finiteness of singular points, Proposition 5.6 implies the assertion.  $\square$

We describe flows near the multi-saddle connection diagram.

**Lemma 5.9.** *Let  $\gamma$  be a multi-saddle connection. If there is a point  $x \notin \gamma$  with  $\omega(x) \cap \gamma \neq \emptyset$ , then the  $\omega$ -limit set  $\omega(x)$  is an attracting limit circuit with  $x \in \text{int}P$  and  $\omega(x) \subseteq \overline{\text{int}P} \cap \gamma$ . On the other hand, if there is a directed circuit  $\mu \subseteq \gamma$  with an associated collar  $\mathbb{A}$  such that  $\overline{O} \cap \mathbb{A} \cap \mu = \emptyset$  for any orbit  $O \subseteq \mathbb{A}$ , then there are periodic orbits in  $\mathbb{A}$  which are essential in  $\mathbb{A}$  and converge to  $\mu$ .*

*Proof.* Suppose that there is a point  $x \notin \gamma$  with  $\omega(x) \cap \gamma \neq \emptyset$ . Since  $|\text{Sing}(v)| < \infty$ , there is an open neighborhood  $U$  of  $\gamma$  such that  $(U - \gamma) \cap \text{Sing}(v) = \emptyset$ . Recall that  $\gamma$  is a compact 1-dimensional cell complex. By  $\omega(x) \cap \gamma \neq \emptyset$ , since each singular point in  $\gamma$  is a multi-saddle, the intersection  $C := \omega(x) \cap \gamma$  is a 1-dimensional cell complex without boundary. The flow box theorem implies that there is an open annulus  $\mathbb{A}$  which consists of finitely many flow boxes as in Figure 6 such that  $C$  is a boundary component of  $\mathbb{A}$ . Since  $\omega(x) \cap \gamma \neq \emptyset$ , the  $\omega$ -limit set of a point in  $\mathbb{A}$  is the attracting limit circuit  $C$ . This means that  $x \in \text{int}P$  and so that  $\omega(x) = C \subseteq \overline{\text{int}P} \cap \gamma$  is an attracting limit circuit. Conversely, suppose there is a directed circuit  $\mu \subseteq \gamma$  with an associated collar  $\mathbb{A}$  such that  $\overline{O} \cap \mathbb{A} \cap \mu = \emptyset$  for any orbit  $O \subseteq \mathbb{A}$ . The flow box

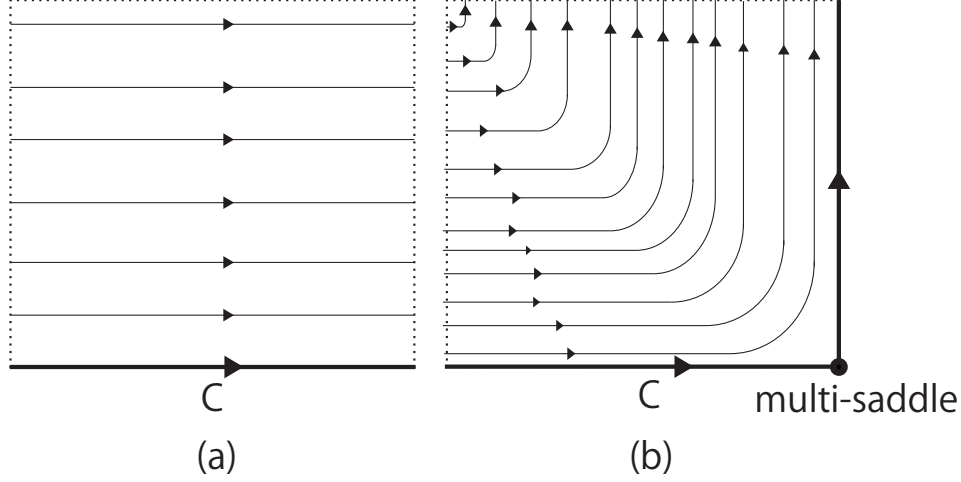


FIGURE 6. (a) A flow box near a regular point in  $C$ . (b) A flow box near a multi-saddle in  $C$ .

theorem implies that we can choose that the annulus  $\mathbb{A}$  consists of finitely many flow boxes as in Figure 6. Then  $\mathbb{A}$  contains no singular points. Fix a regular point  $x \in \mu$  and an transverse arc  $T \subset \overline{\mathbb{A}}$  from  $x$ . Since  $\overline{O \cap \mathbb{A}} \cap \mu = \emptyset$  for any orbit  $O \subseteq \mathbb{A}$ , the first return map with respect to  $T - \{x\}$  has fixed points which converge to  $x$ . Since each fixed point corresponds to a periodic orbit which is essential in  $\mathbb{A}$ , this implies the assertion.  $\square$

**Lemma 5.10.**  $\sigma O \subseteq \delta \text{Cl}(v) \sqcup \delta P \sqcup P_{lc} \sqcup E$  for an orbit  $O \subseteq P$ .

*Proof.* Fix an orbit  $O \subseteq P$ . Since LD is open, by a generalization of the Poincaré-Bendixson theorem, the  $\omega$ -limit (resp.  $\alpha$ -limit) set of  $O$  is either a singular point, a limit circuit, or an exceptional quasi-minimal set. Therefore  $\sigma O = \overline{O} - O \subseteq \delta \text{Cl}(v) \sqcup \delta P \sqcup P_{lc} \sqcup E$ .  $\square$

We show that the height of the flow with finitely many singular points is at most three.

**Proposition 5.11.**  $\text{ht}(S) \leq 3$ .

*Proof.* Since  $\text{ht}(\text{Cl}(v)) = 0$ , Proposition 5.6 implies  $\text{ht}(\delta P \sqcup P_{lc}) = 1$ . By Lemma 4.3, we obtain  $\text{ht}(\text{LD} \sqcup E) \leq 2$ . Lemma 5.10 implies  $\text{ht}(\text{int}P) \leq 3$ . This means  $\text{ht}(S) \leq 3$ .  $\square$

**Lemma 5.12.**  $E \subseteq \overline{P} - P \subseteq E \sqcup \text{Sing}(v) \sqcup \delta \text{Per}(v)$  (i.e.  $\delta P \sqcup E \subseteq \partial P \subseteq \delta P \sqcup E \sqcup \text{Sing}(v) \sqcup \delta \text{Per}(v)$ ).

*Proof.* By Lemma 2.3 [17], we obtain  $E \subseteq \overline{\text{int}P}$ . Recall that  $S = \text{Sing}(v) \sqcup \text{Per}(v) \sqcup P \sqcup \text{LD} \sqcup E$ . Since the union LD and the interior  $\text{int}P$  are open, we have  $\overline{P} - (P \sqcup E) \subseteq \text{Sing}(v) \sqcup \delta \text{Per}(v)$ .  $\square$

Recall the shell  $\sigma A = \overline{A} - A$  of a subset  $A \subseteq S$ . Summarize the differences as follows.

**Proposition 5.13.** *The following statements hold:*

1. LD is open (i.e.  $\partial LD = \sigma LD$ ).
2.  $\sigma \text{Sing}(v) = \emptyset$ .
3.  $\sigma \text{Per}(v) \sqcup \sigma LD \sqcup \sigma E \subseteq \text{Sing}(v) \sqcup \delta P$ .
4.  $\sigma P \subseteq \text{Sing}(v) \sqcup \delta \text{Per}(v) \sqcup E$ .
5.  $E \subseteq \sigma P \cap \text{int}(E \sqcup \text{int} P)$ .

Recall that  $\delta P$  consists of connecting separatrices. Describe the border point set  $\text{Bd} = \partial \text{Sing}(v) \cup \partial \text{Per}(v) \cup \partial P \cup \partial LD \cup \partial E \cup P_{\text{sep}} \cup \partial_{\text{Per}}$  as follows.

**Lemma 5.14.**  $\text{Bd} = \text{Sing}(v) \sqcup \delta \text{Per}(v) \sqcup \delta P \sqcup E \sqcup P_{\text{sep}} \sqcup \partial_{\text{Per}}$ .

*Proof.* The finiteness of  $\text{Sing}(v)$  implies  $\partial \text{Sing}(v) = \text{Sing}(v)$ . Since LD is open, we have  $\delta LD = \emptyset$ . Since there are at most finitely many quasi-minimal sets, Proposition 5.13 implies that  $\partial LD \subseteq \text{Sing}(v) \sqcup \delta P$  and that  $\partial E = \overline{E} \subseteq \text{Sing}(v) \sqcup \delta P \sqcup E$ . Lemma 3.1 implies  $\partial \text{Per}(v) \subseteq \text{Sing}(v) \sqcup \delta \text{Per}(v) \sqcup \delta P$ . Lemma 5.12 implies  $E \subseteq \partial P \subseteq \delta P \sqcup E \sqcup \text{Sing}(v) \sqcup \delta \text{Per}(v)$ . Therefore  $\text{Bd} = \partial \text{Sing}(v) \cup \partial \text{Per}(v) \cup \partial P \cup \partial LD \cup \partial E \cup P_{\text{sep}} \cup \partial_{\text{Per}} = \text{Sing}(v) \sqcup \delta \text{Per}(v) \sqcup \delta P \sqcup E \sqcup P_{\text{sep}} \sqcup \partial_{\text{Per}}$ .  $\square$

**Lemma 5.15.** *The complement  $S - \text{Bd}$  is open such that  $S - \text{Bd} = (\text{int Per}(v) - \partial_{\text{Per}}) \sqcup (\text{int} P - P_{\text{sep}}) \sqcup LD$  and  $LD/\hat{v}$  is a finite space.*

*Proof.* Since  $S$  is compact, there are finitely many boundary components and so the union  $\partial_{\text{Per}}$  consists of finitely many orbits. Since multi-saddles are finite, the union  $P_{\text{sep}}$  consists of finitely many orbits and so  $\overline{\partial_{\text{Per}} \sqcup P_{\text{sep}}} \subseteq \text{Bd}$ .  $\square$

## 5.2. Connecting separatrices and limit circuits.

**Lemma 5.16.** *The following are equivalent for a connecting separatrix  $O$ :*

- 1)  $O \subseteq \text{int} P$ .
- 2)  $O \cap \overline{\text{Per}(v) \sqcup LD \sqcup E} = \emptyset$ .

*Proof.* If  $O \cap \overline{\text{Per}(v) \sqcup LD \sqcup E} = \emptyset$ , then the regularity of  $O$  implies that  $O \subseteq \text{int} P$ . The condition  $O \subseteq \text{int} P$  implies  $O \cap \overline{\text{Per}(v) \sqcup LD \sqcup E} = \emptyset$ .  $\square$

**Lemma 5.17.** *An orbit  $O \subseteq P$  is contained in  $\text{int} P$  if and only if one of the following conditions holds:*

- 1)  $\overline{O} \cap E \neq \emptyset$ .
- 2)  $\overline{O} - O$  contains a limit circuit.
- 3)  $O$  is a connecting separatrix with  $O \cap \overline{\text{Per}(v) \sqcup LD \sqcup E} = \emptyset$ .

*Proof.* Suppose that  $O \subseteq \text{int} P$ . Then  $O \cap \overline{\text{Per}(v) \sqcup LD \sqcup E} = \emptyset$ . By Lemma 5.10, we have  $\overline{O} - O \subseteq \text{Sing}(v) \sqcup \delta \text{Per}(v) \sqcup \delta P \sqcup P_{\text{lc}} \sqcup E$ . Assume that  $\overline{O} \cap E = \emptyset$ . Then  $\overline{O} \subseteq P \sqcup \text{Sing}(v) \sqcup \delta \text{Per}(v)$ . Assume that  $\overline{O} - O$  contains no limit circuits. Since  $\overline{O}$  intersects no quasi-minimal set, each of  $\omega(O)$  and  $\alpha(O)$  is a singular point. Conversely, suppose that  $\overline{O} \cap E \neq \emptyset$ . Lemma 3.6 implies  $O \subseteq \text{int}(P \sqcup E)$ . Since  $\text{int}(P \sqcup E) \subseteq \text{int} P \cup (\partial P \cap \partial E) \cup \text{int} E \subset \text{int} P \cup E$  and since  $O \subseteq P$ , we obtain  $O \subseteq \text{int} P$ . Suppose that  $\overline{O} - O$  contains a limit circuit. Lemma 5.9 implies  $O \subseteq \text{int} P$ . Finally, suppose that  $O$  is a connecting separatrix with  $O \cap \overline{\text{Per}(v) \sqcup LD \sqcup E} = \emptyset$ . Then  $O \subseteq \text{int}(\text{Sing}(v) \sqcup P) = \text{Sing}(v) \sqcup \text{int} P$ . Since  $O$  is not singular, we have  $O \subseteq \text{int} P$ .  $\square$

**Corollary 5.18.** *Each orbit in  $\overline{\text{Per}(v)} \setminus \text{Cl}(v)$  is a connecting separatrix.*



*Proof.* By Lemma 3.1, we have  $\overline{\text{Per}(v)} \setminus \text{Cl}(v) = \partial \text{Per}(v) \cap \delta P$ . Proposition 5.6 implies the assertion holds.  $\square$

### 5.3. Quasi-regular cases.

**Lemma 5.19.** *Suppose that  $v$  is quasi-regular. Each point in  $\overline{\text{Per}(v)} \cap \text{Sing}(v)$  is either a center or a multi-saddle.*

*Proof.* Fix a point  $x$  in  $\overline{\text{Per}(v)} \cap \text{Sing}(v)$ . The quasi-regularity implies that  $x$  is either a center, a  $(\partial^-)$ -sink, a  $(\partial^-)$ -source, or a multi-saddle. For a singular point which is a  $(\partial^-)$ -sink, or a  $(\partial^-)$ -source, there is a neighborhood of it which does not intersect  $\text{Per}(v)$ . Therefore  $x$  is neither a  $(\partial^-)$ -sink, nor a  $(\partial^-)$ -source but so is either a center or a multi-saddle.  $\square$

**Lemma 5.20.** *If  $v$  is quasi-regular, then  $\delta P = D \setminus \text{Cl}(v)$  and  $\delta P \sqcup P_{\text{sep}} = D_{\text{ss}} \setminus (\text{Cl}(v) \sqcup E)$ , where  $D$  (resp.  $D_{\text{ss}}$ ) is the multi-saddle (resp. ss-multi-saddle) connection diagram.*

*Proof.* Let  $\mathbb{S}_+$  be the set of sources and  $\partial$ -sources,  $\mathbb{S}_-$  be the set of sinks and  $\partial$ -sinks,  $W_+ := \bigcup_{x \in \mathbb{S}_+} W^u(x)$ , and  $W_- := \bigcup_{x \in \mathbb{S}_-} W^s(x)$ . Since  $P \cap (W_+ \cup W_-) \subset \text{int} P$ , we have  $\delta P \cap (W_+ \cup W_-) = \emptyset$ . The quasi-regularity implies that each of the  $\omega$ -limit and the  $\alpha$ -limit set of a point in  $\delta P$  is a multi-saddle. Therefore  $\delta P = D \setminus \text{Cl}(v)$ . Recall that  $D_{\text{ss}}$  is the union of multi-saddles, multi-saddle separatrices, ss-separatrices, and ss-components. Then the difference  $D_{\text{ss}} \setminus (\text{Cl}(v) \sqcup E)$  is the union of multi-saddle separatrices and ss-separatrices. This means that  $D_{\text{ss}} \setminus (\text{Cl}(v) \sqcup E) = \delta P \sqcup P_{\text{sep}}$ .  $\square$

## 6. ON THE BORDER POINT SET

Let  $v$  be a flow on a compact connected surface  $S$ . Define an relation  $\sim$  on  $S$  as follows:  $x \sim_{\text{ex}} y$  if either  $O(x) = O(y)$  or both  $x$  and  $y$  are contained in a saddle connection. Write the extended orbit space  $S/v_{\text{ex}} := S/\sim_{\text{ex}}$ . In other words,  $S/v_{\text{ex}}$  can be obtain from  $S/v$  collapsing the saddle connections into singletons. Note that  $(S - \text{Bd})/v = (S - \text{Bd})/v_{\text{ex}}$  and that each class of  $\sim_{\text{ex}}$  is either an orbit or a saddle connection. Denote by  $S/\hat{v}_{\text{ex}}$  the  $T_0$ -identification of  $S/v_{\text{ex}}$ , called the extended orbit class space. In other words, define an relation  $\approx$  on  $S$  as follows:  $x \approx_{\text{ex}} y$  if either  $\overline{O(x)} = \overline{O(y)}$  or both  $x$  and  $y$  are contained in a saddle connection. Then  $S/\hat{v}_{\text{ex}} := S/\approx_{\text{ex}}$ . By Proposition 2.2 [17], the definitions of the equivalences  $\sim_{\text{ex}}$  and  $\approx_{\text{ex}}$  imply the following lemma.

**Lemma 6.1.** *Let  $v$  be a flow on a compact connected surface  $S$ . The following are equivalent:*

- 1)  $S/\hat{v}_{\text{ex}} = S/v_{\text{ex}}$ .
- 2)  $S/\hat{v} = S/v$ .
- 3)  $\text{LD} \sqcup E = \emptyset$ .

We obtain the following statements.

**Lemma 6.2.** *Let  $v$  be a quasi-regular flow on a compact connected surface  $S$ . The following statements hold:*

- 1) *If  $E \subseteq \min S$ , then  $\text{Bd}/\hat{v}_{\text{ex}}$  is an abstract multi-graph whose vertices are centers, periodic orbits on  $\partial S$ , saddle connections, and ss-components except limit circuits, and whose edges are ss-separatrices.*
- 2)  *$\text{Bd}/\hat{v}_{\text{ex}}$  is finite if and only if there are at most finitely many limit cycles.*
- 3)  *$\text{Bd}/\hat{v}_{\text{ex}} = \text{Bd}/v_{\text{ex}}$  if and only if  $E = \emptyset$ .*

*Proof.* Lemma 5.14 implies that  $\text{Bd} = \text{Sing}(v) \sqcup \delta \text{Per}(v) \sqcup \delta \text{P} \sqcup \text{E} \sqcup \text{P}_{\text{sep}} \sqcup \partial \text{Per}$ . By the definition of  $\approx_{\text{ex}}$ , the quasi-regularity implies the assertion 1) holds. By the quasi-regularity, the separatrices are finite. Since there are at most finitely many quasi-minimal sets, a subset  $(\text{Bd} - \delta \text{Per}(v))/\hat{v}_{\text{ex}}$  is finite. This implies the assertion 2). Since  $\text{Bd} \cap \text{LD} = \emptyset$ , the definition of ss-component implies the assertion 3).  $\square$

The finite type condition implies the finiteness of the border point set.

**Theorem 6.3.** *For a flow  $v$  of finite type on a compact connected surface, the quotient space  $\text{Bd}/v_{\text{ex}}$  is a finite abstract multi-graph.*

## 7. DENSITY OF PERIODIC ORBITS IN THE OMEGA LIMIT SET

In this section, we consider when the omega limit set corresponds to the closure of the union of closed orbits. It is known that the rotation number of a circle homeomorphism  $f : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  is rational if and only if  $\overline{\text{Per}(f)} = \Omega(f)$ , where  $\text{Per}(f)$  is the set of periodic points of  $f$ . Note  $\text{Fix}(f) \subseteq \text{Per}(f)$ , where  $\text{Fix}(f)$  is the set of fixed points of  $f$ .

First, we show that the condition  $\overline{\text{Per}(v)} = \Omega(v)$  implies the non-existence of quasi-minimal sets and of strict limit non-periodic circuits. Recall that a non-wandering flow on a compact surface has no exceptional orbits (i.e.  $\text{E} = \emptyset$ ).

**Lemma 7.1.** *Let  $v$  be a flow on a compact surface  $S$ . If  $\overline{\text{Cl}(v)} = \Omega(v)$ , then  $\text{LD} \sqcup \text{E} = \emptyset$  (i.e.  $S = \text{Cl}(v) \sqcup \text{P}$ ).*

*Proof.* Since weakly recurrent orbits are non-wandering, we have  $\text{LD} \sqcup \text{E} \subseteq \Omega(v) = \overline{\text{Cl}(v)}$ . Proposition 5.13 implies that  $\overline{\text{Cl}(v)} \cap (\text{LD} \sqcup \text{E}) = \emptyset$  and so that  $\text{LD} \sqcup \text{E} = \emptyset$ .  $\square$

**Lemma 7.2.** *Let  $v$  be a flow on a compact surface  $S$ . If  $\overline{\text{Cl}(v)} = \Omega(v)$ , then there are no strict limit non-periodic circuit.*

*Proof.* Suppose that there is a strict limit non-periodic circuit  $\gamma$ . The previous lemma implies that  $\gamma$  contains a point in  $\Omega(v) \cap \text{intP} = \Omega(v) - \overline{\text{Cl}(v)}$  and so  $\overline{\text{Cl}(v)} \neq \Omega(v)$ .  $\square$

Note that the converse of Lemma [?] is not true. In other words, there is a flow with uncountably many singular points on a compact surface  $S$  with  $\overline{\text{Cl}(v)} \neq \Omega(v)$ . In fact, consider a Denjoy diffeomorphism  $f : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  with an exceptional minimal set  $\mathcal{M}$ . Let  $v_f$  be the suspension of  $f$  on the torus  $\mathbb{T}^2 := (\mathbb{S}^1 \times \mathbb{R})/(x, r) \sim (f(x), r + 1)$  and  $\widetilde{\mathcal{M}}$  the minimal set of  $v_f$ . Fix a bump function  $\varphi : \mathbb{T}^2 \rightarrow \mathbb{R}_{\geq 0}$  with  $\varphi^{-1}(0) = \widetilde{\mathcal{M}}$ , where  $\hat{\mathcal{M}} := \widetilde{\mathcal{M}} \cap (\mathbb{S}^1 \times \{1/2\})$  is a lift of  $\mathcal{M}$ . Using the bump function  $\varphi$ , replace the minimal set  $\widetilde{\mathcal{M}}$  of  $v_f$  with a union of singular points and separatrices of the resulting flow  $v_\varphi$  (i.e.  $\widetilde{\mathcal{M}} = \text{Sing}(v_\varphi) \sqcup \{\text{separatrix of } v_\varphi\}$ ) such that  $O_{v_\varphi}(x) = O_{v_f}(x)$  for any point  $x \in \mathbb{T}^2 - \widetilde{\mathcal{M}}$ . Then  $\mathbb{T}^2 = \text{Sing}(v_\varphi) \sqcup \text{P}(v_\varphi)$  and  $\overline{\text{Cl}(v_\varphi)} = \text{Sing}(v_\varphi) \neq \widetilde{\mathcal{M}} = \Omega(v_\varphi)$ , where  $\text{P}(v_\varphi)$  is the union of non-closed proper orbits of  $v_\varphi$  and  $\Omega(v_\varphi)$  is the set of non-wandering points. Moreover, there is a flow  $v$  on a compact surface  $S$  with countable singular points and with a non-periodic non-limit directed circuit in  $\Omega(v)$  such that  $\overline{\text{Cl}(v)} \neq \Omega(v)$ . Indeed, consider a toral flow  $w$  with one limit cycle  $C$  but without singular points. Fix a point  $z \in C$  and a non-closed proper orbit  $O$ . Write an open disk  $D := \mathbb{T}^2 - (C \sqcup O)$ . Choose a closed transversal  $T$  through  $z$ , a point  $x \in O$ , and a sequence  $(t_n)_{n \in \mathbb{Z}}$

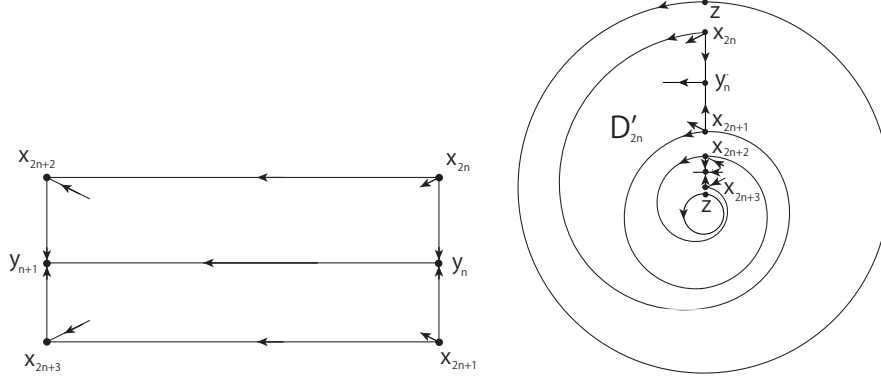


FIGURE 7.

on  $O$  such that  $T \cap O = \{x_n\}_{n \in \mathbb{Z}}$ ,  $\lim_{n \rightarrow \infty} t_n = \infty$ ,  $\lim_{n \rightarrow -\infty} t_n = -\infty$ , and  $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow -\infty} x_n = z$ , where  $x_n := w_{t_n}(x)$ . Write  $D_n$  the open flow box such that  $D - T = \bigcup_{n \in \mathbb{Z}} D_n$ . Write an open flow box  $D'_{2n} = D_{2n} \sqcup D_{2n+1} \sqcup ((T \setminus O) \cap \overline{D_{2n}} \cap \overline{D_{2n+1}}) \subset D$ . Replacing the closure of each  $D'_{2n}$  by a box as Figure 7, we obtain the resulting flow  $v$  on the torus  $\mathbb{T}^2$  such that the singular point set  $\text{Sing}(v) = \{z\} \sqcup \{x_n, y_n\}_{n \in \mathbb{Z}} \subset T$  is countable,  $\mathbb{T}^2 = \text{Sing}(v) \sqcup P$ , and  $\Omega(v) = C \sqcup \{x_n, y_n\}_{n \in \mathbb{Z}} \supsetneq \overline{\text{Cl}(v)}$ .

We describe a property of neighborhoods of directed non-limit cycles.

**Lemma 7.3.** *Let  $v$  be a flow on a compact surface  $S$  and  $\gamma$  a directed circuit with isolated singular points and with an associated collar  $\mathbb{A}$  of  $\gamma$  such that  $\gamma$  is not a limit circuit with respect to  $\mathbb{A}$ . Then  $\gamma \subset \overline{\text{Per}(v)}$ . Moreover there is a sequence  $(O_n)_{n \in \mathbb{Z}_{>0}}$  of periodic orbits each of which is essential in  $\mathbb{A}$  and is homologous to  $\gamma$ .*

*Proof.* Fix a regular point  $x \in \gamma$  and an transverse arc  $T \subset \overline{\mathbb{A}}$  from  $x$ . Since  $\overline{O} \cap \mathbb{A} \cap \gamma = \emptyset$  for any orbit  $O$  intersecting  $\mathbb{A}$ , the first return map with respect to  $T - \{x\}$  has fixed points which converge to  $x$ . Since each fixed point corresponds to a periodic orbit which is essential in  $\mathbb{A}$ , this implies  $\gamma \subset \overline{\text{Per}(v)}$ .  $\square$

We consider the non-wandering case.

**Lemma 7.4.** *Let  $v$  be a flow on a compact surface  $S$ . The following are equivalent:*

- 1)  $v$  is non-wandering and  $\overline{\text{Cl}(v)} = \Omega(v)$ .
- 2)  $v$  is non-wandering and  $\text{LD} = \emptyset$ .
- 3)  $\text{int}P = \emptyset$  and  $\overline{\text{Cl}(v)} = \Omega(v)$ .
- 4)  $\text{LD} \sqcup \text{int}P = \emptyset$  (i.e.  $S = \text{Cl}(v) \sqcup \delta P$ ).
- 5)  $\overline{\text{Cl}(v)} = S$ .

*Proof.* Obviously, the conditions 1) and 5) are equivalent. Theorem 3.2 [18] implies that the non-wandering property is equivalent to the property  $\text{int}P = \emptyset$ . This implies that the conditions 1) and 3) (resp. 2) and 4)) are equivalent. We show that the condition 4) is equivalent to the condition 5). Indeed, suppose  $\text{LD} \sqcup \text{int}P = \emptyset$ . By Lemma 2.3 [17], we have  $E = \emptyset$  and so  $S = \text{Cl}(v) \sqcup P$ . Since  $\text{int}P = \emptyset$ , we have  $S = \overline{\text{Cl}(v)}$ . Conversely, suppose  $\overline{\text{Cl}(v)} = S$ . Then  $\text{Sing}(v) \cup \overline{\text{Per}(v)} = S$ . By

Lemma 3.1, we have  $\overline{LD} \subseteq \overline{\text{Sing}(v) \sqcup \delta P \sqcup LD} \subseteq S = \text{Sing}(v) \cup \overline{\text{Per}(v)}$  and so  $\delta P \sqcup LD \subseteq \overline{\text{Per}(v)}$ . Since  $\overline{\text{Per}(v)} \cap LD = \emptyset$ , we obtain  $LD = \emptyset$ . This implies that  $LD \sqcup \text{int}P = \emptyset$ .  $\square$

We consider the case  $S = \text{Per}(v) \sqcup P$ .

**Lemma 7.5.** *Let  $v$  be a flow on a compact surface  $S$ . If  $S = \text{Per}(v) \sqcup P$ , then  $P$  is open and  $\Omega(v) = \overline{\text{Cl}(v)} = \text{Per}(v)$ .*

*Proof.* Since each omega-limit (resp. alpha-limit) set of an orbit in  $P$  is a limit cycle, the union  $P$  is open and so  $\Omega(v) = \overline{\text{Cl}(v)} = \text{Per}(v)$ .  $\square$

**7.1. Density of periodic orbits in the omega limit set for (quasi-)regular flows.** We consider the regular case without  $(\partial)$ -saddles (i.e. quasi-regular case without multi-saddles).

**Lemma 7.6.** *Let  $v$  be a quasi-regular flow without multi-saddles on a compact surface  $S$ . Then  $LD \sqcup E = \emptyset$  if and only if  $\overline{\text{Cl}(v)} = \Omega(v)$ . In each case, we have  $\text{Cl}(v) = \Omega(v)$ .*

*Proof.* By Lemma 7.1, the condition  $\overline{\text{Cl}(v)} = \Omega(v)$  implies  $LD \sqcup E = \emptyset$ . Conversely, suppose that  $LD \sqcup E = \emptyset$ . Taking the orientation double covering (resp. the double of a manifold), we may assume that  $S$  is closed orientable. The Poincaré-Hopf theorem implies that  $S$  is either a sphere or a torus. The quasi-regularity and the hypothesis imply that each singular point is either a center, a sink, or a source. Note each index of singular point is one. Suppose that  $S$  is spherical. Then  $S = \text{Sing}(v) \sqcup \text{Per}(v) \sqcup P$ . The Poincaré-Hopf theorem implies that there are exactly two singular points. We show that the union  $\text{Cl}(v)$  is closed. Indeed, assume that there are two singular points which are sinks or sources. Define  $\sigma$  (resp.  $\mu$ ) as follows:  $\sigma = u$  if  $x$  is a sink (resp.  $\mu = u$  if  $y$  is a sink) and  $\sigma = s$  if  $x$  is a source (resp.  $\mu = s$  if  $y$  is a source). Then  $W^\sigma(x)$  (resp.  $W^\mu(y)$ ) is an open disk and  $W^\sigma(x) - \{x\}, W^\mu(y) - \{y\} \subseteq \text{int}P$ . Then each of the boundaries  $\partial W^\sigma(x)$  and  $\partial W^\mu(y)$  is a limit cycle. Applying Lemma 7.5 to the compact surface  $S - (W^\sigma(x) \sqcup W^\mu(y))$ , the union  $P$  is open and so  $\delta P = \emptyset$ . By Lemma 7.4, the union  $\text{Cl}(v)$  is closed. Assume that  $v$  has two centers. Then the complement of two centers consists of regular points in  $\text{Per}(v) \sqcup P$ . Fix two disjoint center disks  $D, D'$  each of whose boundary is a periodic orbit. Applying Lemma 7.5 to the compact surface  $S - (D \sqcup D')$ , the union  $P$  is open and so  $\delta P = \emptyset$ . By Lemma 7.4, the union  $\text{Cl}(v)$  is closed. Thus we may assume there is exactly one center. Then there is a sink or a source. By the time reversion if necessary, we may assume that there is a sink  $x$ . Then  $W^\sigma(x) - \{x\} \subseteq \text{int}P$ . Fix a center disks  $D$  whose boundary is a periodic orbit. Applying Lemma 7.5 to the compact surface  $S - (D \sqcup W^\sigma(x))$ , the union  $P$  is open and so  $\delta P = \emptyset$ . By Lemma 7.4, the union  $\text{Cl}(v)$  is closed. Then the complement  $P = S - \text{Cl}(v)$  is open and is the set of non-weakly-recurrent points. This means that  $\Omega(v) = \text{Cl}(v) = \overline{\text{Cl}(v)}$ . Suppose that  $S$  is toral. Since  $v$  is regular and has no saddles, the flow has no singular points. Then  $S = \text{Per}(v) \sqcup P$ . By Lemma 7.5, the assertion holds.  $\square$

Note that a regular flow without saddles is a quasi-regular flow without multi-saddles and vice versa.

**Lemma 7.7.** *Let  $v$  be a quasi-regular flow on a connected compact surface  $S$ . Suppose that the multi-saddle connection diagram contains no directed circuit (i.e. is a forest as a graph). Then  $LD \sqcup E = \emptyset$  if and only if  $\overline{Cl(v)} = \Omega(v)$ . In each case, we have  $Cl(v) = \Omega(v)$ .*

*Proof.* By Lemma 7.1, the condition  $\overline{Cl(v)} = \Omega(v)$  implies  $LD \sqcup E = \emptyset$ . Conversely, suppose that  $LD \sqcup E = \emptyset$ . Then  $S = Cl(v) \sqcup P$ . We may assume that  $S \neq Cl(v)$ . By Lemma 7.6, we may assume that there are multi-saddles. As above, taking the orientation double covering (resp. the double of a manifold), we may assume that  $S$  is closed orientable. Let  $D$  be the multi-saddle connection diagram. We show that  $P$  is open. Indeed, by Lemma 5.1, the quasi-regularity implies each boundary component of  $Per(v)$  is either a center or a limit cycle. This means that  $\partial Per(v) \subseteq Cl(v)$  and so  $Cl(v)$  is closed. Then the complement  $P = S - Cl(v)$  is open. By the characterization of  $\omega$ -limit (resp.  $\alpha$ -limit) sets, the  $\omega$ -limit (resp.  $\alpha$ -limit) set of a point in  $P \setminus D$  is either a limit cycle, a sink, a source, or a multi-saddle. This implies that each orbit closure of a point in  $P \setminus D$  contains either a limit cycle, a sink, or a source. Then there is a closed transversal around either a limit cycle, a sink, or a source which contained in  $P$ . Therefore each point in  $P$  is wandering (i.e.  $P \cap \Omega(v) = \emptyset$ ) and so  $Cl(v) = \Omega(v)$ .  $\square$

To use the induction of the number of directed non-periodic circuits, we show the following lemma.

**Lemma 7.8.** *Let  $v$  be a quasi-regular flow on a compact surface  $S$  and  $\gamma$  a directed circuit. Suppose that there are no strict limit non-periodic circuit. Removing  $\gamma$ , taking the metric completion of  $S - \gamma$ , and collapsing each new boundary component into a singular point, denote by  $w$  the resulting flow on the resulting compact surface  $T$ . Then  $Cl(v) = \Omega(v)$  if and only if  $Cl(w) = \Omega(w)$ .*

*Proof.* Since there are no strict limit circuit which is not a periodic orbit, we have  $\gamma \subset Per(v) \sqcup LD \sqcup E$ . Suppose that  $\overline{Cl(v)} = \Omega(v)$ . Theorem 7.1 implies  $LD \sqcup E = \emptyset$ . By construction, we have  $LD(w) \sqcup E(w) = \emptyset$ . This implies that  $S = Sing(v) \sqcup Per(v) \sqcup P$  and  $T = Sing(w) \sqcup Per(w) \sqcup P(w)$ . For a point  $x \in S - \gamma$ , denote  $\hat{x}$  the resulting point of  $x$  in  $T$ . Fix a point  $x \in S$ . Suppose that  $x \in \Omega(v)$ . If  $x \in \gamma$ , then  $x \in Per(v)$  and so each resulting point of  $x$  is contained in  $Per(w) \subseteq \Omega(w)$ . Thus we may assume that  $x \in \Omega(v) - \gamma$ . For any neighborhood  $\hat{U} \subset T - \hat{\gamma}$  of  $\hat{x}$ , there is an arbitrarily large number  $r > 0$  such that  $U \cap \bigcup_{t>r} v_t(U) \neq \emptyset$ , where  $U := \{x \in S \mid \hat{x} \in \hat{U}\}$  is a neighborhood of  $x$ . Since the circuit  $\gamma$  is closed invariant, the resulting subset  $\hat{U} := \{\hat{x} \mid x \in U \setminus \gamma\}$  is a neighborhood of  $\hat{x}$  such that there is an arbitrarily large number  $r > 0$  such that  $U \cap \bigcup_{t>r} v_t(U) \neq \emptyset$ . This means  $\hat{x} \in \Omega(w)$ . Suppose that  $x \in S - \Omega(v)$ . For any neighborhood  $\hat{U} \subset T - \hat{\gamma}$  of  $\hat{x}$ , there is a large number  $r > 0$  such that  $U \cap \bigcup_{t>r} v_t(U) = \emptyset$ , where  $U := \{x \in S \mid \hat{x} \in \hat{U}\}$  is a neighborhood of  $x$ . Then  $\hat{U} \cap \bigcup_{t>r} w_t(\hat{U}) = \emptyset$ . This means  $\hat{x} \in T - \Omega(w)$ . Conversely, suppose that  $\overline{Cl(w)} = \Omega(w)$ . As above, we obtain  $S = Sing(v) \sqcup Per(v) \sqcup P$  and  $T = Sing(w) \sqcup Per(w) \sqcup P(w)$ . Fix a point  $\hat{x} \in T$ . First suppose that  $\hat{x} \in \Omega(w)$ . If  $\hat{x}$  is a resulting point in  $T$  of  $\hat{\gamma}$ , then the original point is contained in  $\gamma \subset Per(v)$  and so is contained in  $\Omega(v)$ . Thus we may assume that  $\hat{x} \in \Omega(w) - \hat{\gamma}$ . For any neighborhood  $U \subseteq S - \gamma$  of  $x$ , there is an arbitrarily large number  $r > 0$  such that  $\hat{U} \cap \bigcup_{t>r} w_t(\hat{U}) \neq \emptyset$ . Then there is an arbitrarily

large number  $r > 0$  such that  $U \cap \bigcup_{t>r} v_t(U) \neq \emptyset$ . This implies  $x \in \Omega(v)$ . Second suppose that  $\hat{x} \in T - \Omega(w)$ . For any neighborhood  $U \subseteq S - \gamma$  of  $x$ , there is a large number  $r > 0$  such that  $\hat{U} \cap \bigcup_{t>r} w_t(\hat{U}) = \emptyset$ . Then  $U \cap \bigcup_{t>r} v_t(U) = \emptyset$ . This implies  $x \in S - \Omega(v)$ .  $\square$

**Proposition 7.9.** *Let  $v$  be a quasi-regular flow on a compact surface  $S$ . If  $\text{LD} \sqcup \text{E} = \emptyset$  and there are no strict limit non-periodic circuit, then  $\overline{\text{Cl}(v)} = \Omega(v)$ .*

*Proof.* Suppose that  $\text{LD} \sqcup \text{E} = \emptyset$  and there are no strict limit non-periodic circuit. By the characterization of  $\omega$ -limit (resp.  $\alpha$ -limit) sets (cf. Theorem 2.1 [10]), the absence of quasi-minimal sets implies that each  $\omega$ -limit (resp.  $\omega$ -limit) is either a closed orbit or a limit circuit and that  $S = \text{Sing}(v) \sqcup \text{Per}(v) \sqcup \text{P} = \text{Cl}(v) \sqcup \text{P}$ . Since the relation  $\Omega(v) = \overline{\text{Cl}(v)}$  is invariant under the orientation double covering (resp. taking the double of a manifold), we may assume that  $S$  is closed orientable. Let  $N(v)$  be the number of directed circuits. By induction on  $N(v)$ , we show the assertion. Indeed, if  $N(v) = 0$ , then Lemma 7.7 implies the assertion. Thus we may assume that there is a directed circuit  $\gamma$  in the multi-saddle connection diagram. Since there are no strict limit non-periodic circuit, taking an innermost circuit, for any edge  $e$  of  $\gamma$ , there is a directed circuit  $\mu$  with the associated collar  $\mathbb{A}$  such that  $\mu$  contains  $e$  but is not a limit circuit with respect to  $\mathbb{A}$ . By Lemma 7.3, we have  $\mu \subset \overline{\text{Per}(v)}$ . As Lemma 7.8, removing  $\gamma$ , taking the metric completion of  $S - \gamma$ , and collapsing each new boundary into a singular point, denote by  $w$  the resulting flow on the resulting compact surface  $T$ . Then we have  $N(w) < N(v)$ . By Lemma 7.8, the inductive hypothesis implies the assertion.  $\square$

Summarizing the necessary condition and the sufficient condition, we obtain the following statement.

**Theorem 7.10.** *Let  $v$  be a quasi-regular flow on a compact surface  $S$ . The following conditions are equivalent:*

- 1)  $\text{LD} \sqcup \text{E} = \emptyset$  and there are no strict limit non-periodic circuit.
- 2)  $\overline{\text{Cl}(v)} = \Omega(v)$ .

Note the quasi-regularity is necessary. In other words,  $\Omega(v) - \overline{\text{Cl}(v)} \neq \emptyset$  in general. Indeed, since each point in the circle containing  $z$  in Figure 7 is non-wandering, we have  $\Omega(v) - \overline{\text{Cl}(v)} \neq \emptyset$  for the flow  $v$  in Figure 7 if the circle containing  $z$  is contained in the boundary  $\partial S$ .

## 8. STRUCTURES OF ORBITS

Summarize properties of the boundary points and border points as follows.

**Theorem 8.1.** *Let  $v$  be a flow with finitely many singular points on a compact surface  $S$ . The following statements hold:*

- 1)  $\partial \text{Per}(v) \subseteq \text{Sing}(v) \sqcup \delta \text{Per}(v) \sqcup \delta \text{P}$ .
- 2)  $\partial \text{LD} \subseteq \text{Sing}(v) \sqcup \delta \text{P}$ .
- 3)  $\partial \text{E} = \overline{\text{E}} \subseteq \text{Sing}(v) \sqcup \delta \text{P} \sqcup \text{E}$ .
- 4)  $\partial \text{P} \subseteq \text{Sing}(v) \sqcup \delta \text{Per}(v) \sqcup \delta \text{P} \sqcup \text{E}$ .
- 5)  $\delta \text{P}$  is a union of finitely many connecting separatrices.
- 6)  $\delta \text{Per}(v)$  is contained in the closure of the union of limit cycles.
- 7)  $\text{Bd} = \text{Sing}(v) \sqcup \delta \text{Per}(v) \sqcup \delta \text{P} \sqcup \text{E} \sqcup \text{P}_{\text{sep}} \sqcup \partial \text{Per}$ .

*Proof.* Proposition 5.13 implies the assertions 1) - 4). Proposition 5.6 implies the assertion 5). Lemma 3.5 implies the assertion 6). Lemma 5.14 implies the assertion 7).  $\square$

**8.1. The border point set.** Let  $v$  be a flow with finitely many singular points on a connected compact surface  $S$ . The previous theorem implies following descriptions of the border point set  $Bd$  as follows.

**Corollary 8.2.** *Let  $v$  be a quasi-regular flow on a compact surface  $S$ . Denote by  $S_c$  the set of centers, and by  $D_{ss}$  the ss-multi-saddle connection diagram. The following statements hold:*

- 1)  $\delta P \sqcup P_{sep}$  is the union of multi-saddle separatrices and ss-separatrices.
- 2)  $\delta Per(v)$  is contained in the closure of the union of limit cycles.
- 3)  $Bd = D_{ss} \sqcup S_c \sqcup \partial_{Per}$  is closed.
- 4)  $D_{ss} = (Sing(v) - S_c) \sqcup \delta Per(v) \sqcup \delta P \sqcup P_{sep} \sqcup E$  is closed.

The non-existence of exceptional orbits implies the following properties.

**Corollary 8.3.** *The following statements hold for a quasi-regular flow  $v$  with  $E = \emptyset$  on a compact surface  $S$ :*

- 1)  $S - Bd = (int Per(v) - \partial_{Per}) \sqcup int(P - P_{sep}) \sqcup LD$  is open.
- 2) The orbit space  $((int Per(v) - \partial_{Per}) \sqcup int(P - P_{sep}))/v$  is a 1-manifold.
- 3) The orbit class space  $LD/\hat{v}$  is a finite discrete space.
- 4)  $(S - Bd)/\hat{v}$  is a finite union of circles, intervals, and points.

*Proof.* The previous corollary implies the assertion 1). Taking the double of a manifold, we may assume that  $S$  is closed. Since there are at most finitely many quasi-minimal sets, the assertion 3) holds. Since a small saturated neighborhood of a periodic orbit in  $int Per(v)$  is either is an annulus or a Möbius band, The orbit space  $(int Per(v) - \partial_{Per})/v$  is a 1-manifold. By a generalization of the Poincaré-Bendixson theorem, the  $\omega$ -limit (resp.  $\alpha$ -limit) set of a non-closed proper orbit is either a sink, a source, or a limit circuit. Taking a closed transversal which either is parallel to a limit circuit or bounds a sink disk or a source disk, The orbit space  $int(P - P_{sep})/v$  is an interval or a circle.  $\square$

**8.2. Hierarchy of orbits.** To convert the flow  $v$  into discrete data, we state the relations between height and types of orbits.

	$S_k \subseteq$	$S_k \cap \max S \subseteq$	$S_k \setminus \max S \subseteq$
$k = 3$	$int P - P_{lc}$	$int P - P_{lc}$	$\emptyset$
$k = 2$	$(int P - P_{lc}) \sqcup LD \sqcup E$	$(int P - P_{lc}) \sqcup LD$	$E$
$k = 1$	$P \sqcup LD \sqcup E$	$(P - P_{lc}) \sqcup LD$	$\delta P \sqcup P_{lc} \sqcup E$
$k = 0$	$S - P$	$int Cl(v) \sqcup LD$	$\delta Cl(v) \sqcup E$

TABLE 1.

**Corollary 8.4.** *Let  $v$  be a flow with finitely many singular points on a connected compact surface  $S$ . The following statements hold:*

- 1)  $\delta Per(v) \subseteq \min S \setminus \max S \subseteq \delta Cl(v) \sqcup E$ .

	$\subseteq S_k$	$\subseteq S_k \cap \max S$	$\subseteq S_k \setminus \max S$
$k = 3$	$\emptyset$	$\emptyset$	$\emptyset$
$k = 2$	$\emptyset$	$\emptyset$	$\emptyset$
$k = 1$	$\delta P \sqcup P_{lc}$	$\delta P \cap \text{int}(\text{Per}(v) \sqcup \delta P)$	$(\delta P \cap \overline{\text{int}P \sqcup LD \sqcup E}) \sqcup P_{lc}$
$k = 0$	$\text{Cl}(v)$ $\sqcup ((LD \sqcup E) \cap \min S)$	$\text{int Cl}(v)$ $\sqcup (LD \cap \min S)$	$\delta \text{Sing}(v) \sqcup \delta \text{Per}(v)$ $\sqcup (E \cap \min S)$

TABLE 2.

- 2)  $\text{int Cl}(v) = \min S \cap \max S$  if  $v$  is not minimal.
- 3)  $\delta P \sqcup P_{lc} \subseteq S_1$ .
- 4)  $S_1 \sqcup S_2 \subseteq P \sqcup LD \sqcup E$ .
- 5)  $S_3 \subseteq \text{int}P - P_{lc}$ .
- 6)  $\text{int Cl}(v) \sqcup (\text{int}P - P_{lc}) \sqcup LD \subseteq \max S \subseteq \text{Cl}(v) \sqcup (P - P_{lc}) \sqcup LD$ .
- 7)  $P \subseteq S_1 \cup \max S$ .
- 8)  $LD \subseteq (S_0 \sqcup S_1 \sqcup S_2) \cap \max S$ .
- 9)  $E \subseteq S - \max S$ .
- 10)  $\text{ht}(S) \leq 3$ .

*Proof.* Since closed orbits are minimal sets, we have the assertions 1) and 4). By Lemma 3.6, the assertions 2) and 9) hold. Proposition 5.6 implies  $\delta P \subseteq S_1$ . By the definition of a limit circuit, we have  $P_{lc} \subset S_1$ . This mean that the assertion 3) is true. Lemma 5.10 implies the assertion 5). By the definition of  $\max S$ , the assertion 6) holds. Since  $\text{int}P \subseteq \max S$ , Proposition 5.6 implies that the assertion 7). Lemma 5.3 and Lemma 5.8 imply the assertion 8). Proposition 5.11 implies the assertion 10).  $\square$

We list all possibilities of orbit closures of maximal points with respect to the pre-order.

**Lemma 8.5.** *Let  $x \in S_3$  be a point. Then  $\overline{O(x)} \cap S_2 \cap E \neq \emptyset$ . Moreover, if  $\omega(x) \cap S_2 \cap E \neq \emptyset$ , then the following statements hold:*

$$\begin{aligned}
O(x) &= \overline{O(x)} \cap S_3 \subseteq \text{int}P - P_{lc} \\
\omega(x) \cap S_2 &\subseteq E \\
\omega(x) \cap S_1 &\subseteq \delta P \\
\omega(x) \cap S_0 &\subseteq \text{Sing}(v)
\end{aligned}$$

**Lemma 8.6.** *Let  $x \in S_2 \cap \max S$  be a point.*

*If  $x \in LD$ , then the following statements hold:*

$$\begin{aligned}
\hat{O}(x) &= \overline{O(x)} \cap S_2 \subseteq LD \\
\overline{O(x)} \cap S_1 &\subseteq \delta P \\
\overline{O(x)} \cap S_0 &\subseteq \text{Sing}(v)
\end{aligned}$$



	ht = 3	ht = 2	ht = 1	ht = 0
$E + \delta P$	$\text{int}P - P_{lc}$	$E$	$\delta P$	$\delta \text{Sing}(v)$
$E + \text{Sing}(v)$		$\text{int}P - P_{lc}$	$E$	$\delta \text{Sing}(v)$
$E$			$\text{int}P - P_{lc}$	$E$
$LD + \delta P$		$LD$	$\delta P$	$\delta \text{Sing}(v)$
$LD + \text{Sing}(v)$			$LD$	$\delta \text{Sing}(v)$
$LD$				$LD$
Limit circuit		$\text{int}P - P_{lc}$	$\delta P \sqcup P_{lc}$	$\delta \text{Sing}(v)$
Limit cycle			$\text{int}P - P_{lc}$	$\delta \text{Per}(v)$
Separatrix			$P - P_{lc}$	$\delta \text{Sing}(v)$
$\text{int Cl}(v)$				$\text{int Cl}(v)$

TABLE 3.

If  $x \in \text{int}P - P_{lc}$  and  $\overline{O(x)} \cap E = \emptyset$ , then  $\overline{O(x)}$  contains a limit circuit and the following statements hold: If  $\omega(x)$  is a limit circuit, then

$$\begin{aligned} O(x) &= \overline{O(x)} \cap S_2 \subseteq \text{int}P - P_{lc} \\ \omega(x) \cap S_1 &\subseteq \delta P \sqcup P_{lc} \\ \omega(x) \cap S_0 &\subseteq \text{Sing}(v) \end{aligned}$$

If  $\omega(x) \cap E \neq \emptyset$  (and so  $x \in \text{int}P - P_{lc}$ ), then the following statements hold:

$$\begin{aligned} O(x) &= \overline{O(x)} \cap S_2 \subseteq \text{int}P - P_{lc} \\ \omega(x) \cap S_1 &\subseteq E \\ \omega(x) \cap S_0 &\subseteq \text{Sing}(v) \end{aligned}$$

**Lemma 8.7.** Let  $x \in S_1 \cap \max S$  be a point.

If  $x \in LD$ , then the following statements hold:

$$\begin{aligned} \hat{O}(x) &= \overline{O(x)} \cap S_1 \subseteq LD \\ \overline{O(x)} \cap S_0 &\subseteq \text{Sing}(v) \end{aligned}$$

If  $x \in \delta P$ , then the following statements hold:

$$\begin{aligned} O(x) &= \overline{O(x)} \cap S_1 \subseteq \delta P \\ \overline{O(x)} \cap S_0 &\subseteq \text{Sing}(v) \end{aligned}$$

If  $x \in \text{int}P - P_{lc}$ , then the following statements hold:

$$\begin{aligned} O(x) &= \overline{O(x)} \cap S_1 \subseteq \text{int}P \\ \overline{O(x)} \cap S_0 &\subseteq \text{Cl}(v) \end{aligned}$$

**Lemma 8.8.**  $\min S \cap \max S \subseteq \text{int Cl}(v) \sqcup LD$ .

## 9. DECOMPOSITIONS OF FLOWS

Cutting essential parts, we reduce surface flows into spherical flows.

**Lemma 9.1.** *Let  $v$  is a quasi-regular flow on a compact surface  $S$ ,  $S_o$  the resulting surface from  $S$  by applying an operation  $C_o$  as possible,  $S_d$  the resulting surface from  $S_o$  by applying an operation  $C_d$ , and  $S_t$  the resulting surface from  $S_d$  by applying an operation  $C_t$  as possible. Then the surface  $S_t$  is a subset of a sphere.*

*Proof.* Removing 0-saddles and  $\partial$ -0-saddles, we may assume that there are neither 0-saddles nor  $\partial$ -0-saddles. Let  $v_\sigma$  be the resulting flow from  $v$  on  $S_\sigma$ , where  $\sigma \in \{o, d, t\}$ . Since  $S$  is compact, the surface  $S_o$  can be obtain by operations  $C_o$  finitely many times. By construction, the flow  $v_o$  has no essential periodic orbits. By construction, the flow  $v_d$  has neither essential periodic orbits nor essential loops in  $D$ . Since  $S_d$  is compact, the surface  $S_t$  can be obtain by operations  $C_t$  finitely many times. Then the flow  $v_t$  has neither essential closed transversals, essential loops in  $D$ , nor essential periodic orbits. Since each quasi-minimal set intersects some essential closed transversal, the flow  $v_t$  has no quasi-minimal sets but consists of proper orbits. We show that  $S_t$  is a subset of a sphere. Indeed, replacing  $\partial$ -sinks (resp.  $\partial$ -sources) with pairs of a  $\partial$ -saddle and a sink (resp. a source) (see Figure 8), we may assume that there are neither  $\partial$ -sinks nor  $\partial$ -sources on  $S_t$ . Then each singular point on the boundary  $\partial S$  is a multi-saddle. Let  $S_3$  be the resulting closed surface from  $S_t$  by collapsing all boundaries into singletons and  $v_3$  the resulting flow on  $S_3$ . Then each new singular point of  $v_3$  is a multi-saddle. Note that the surface  $S_t$  is a subset of a sphere if and only if each connected component of  $S_3$  is a sphere. Therefore we may assume that  $S_3$  is connected. The quasi-regularity of  $v$  implies that each singular point of the resulting flow  $v_3$  is either a multi-saddle, a sink, a source, or a center. Recall that each separatrix connecting a sink or a source does not belong to any multi-saddle connection. For a  $k$ -saddle connection of  $v_3$  ( $k > 0$ ) such that there is a separatrix from a source to it (resp. from it to a sink), applying the inverse operation of  $\text{Ch}_l$  to the separatrix, we obtain a  $(k - 1)$ -saddle connection (see Fig. 9). Applying the operations finitely many times and removing 0-saddles, we may assume that there are no separatrices connecting a multi-saddle and either a sink or a source. Then each separatrix either connects a sink and a source or is contained in the multi-saddle connection diagram  $D(v_3)$  of  $v_3$ . By Lemma 5.9, the directed circuit is a limit circuit or is parallel to a periodic orbit. This means that each multi-saddle connection of  $v_3$  is null homologous with respect to  $H_1(S, \partial S)$ . Since there are no boundaries of  $S_3$ , each multi-saddle connection of  $v_3$  is null homotopic. Collapsing closed disks each of whose boundaries is contained in the multi-saddle connection diagram  $D(v_3)$  into singletons, we may assume that  $D(v_3) \subseteq \partial S$  and that each periodic orbit is null homotopic. Therefore  $v_3$  consists of sinks, sources, centers, non-closed proper orbits, and null homotopic periodic orbits. Since there are no essential periodic orbits, there are singular points whose indices are positive. Poincaré-Hopf theorem implies that  $S_3$  is a sphere. Therefore  $S_t$  is a subset of a sphere.  $\square$

Note that the operation  $C_d$  is necessary in the previous lemma (see Figure 10). Recall that  $\text{Bd} = \partial \text{Sing}(v) \cup \partial \text{Per}(v) \cup \partial \text{P} \cup \partial \text{LD} \cup \partial \text{E} \cup \text{P}_{\text{sep}} \cup \partial \text{Per}$ . We describe the complement of the border point set  $\text{Bd}$  as follows.

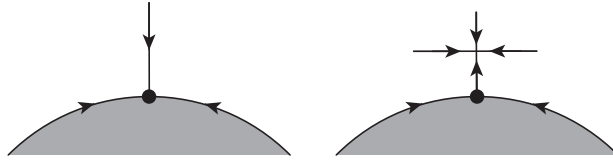
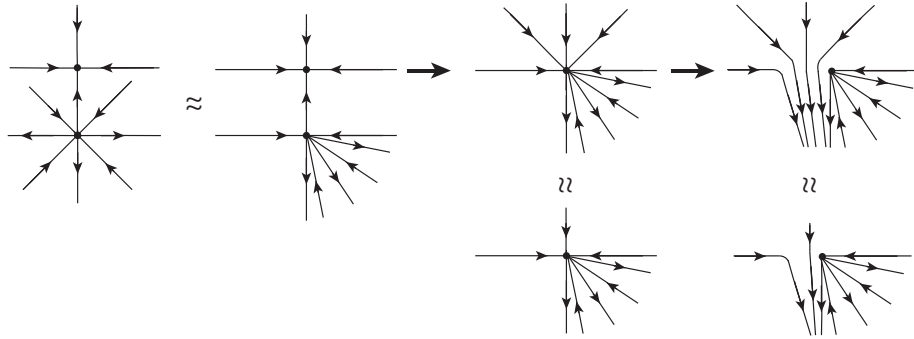
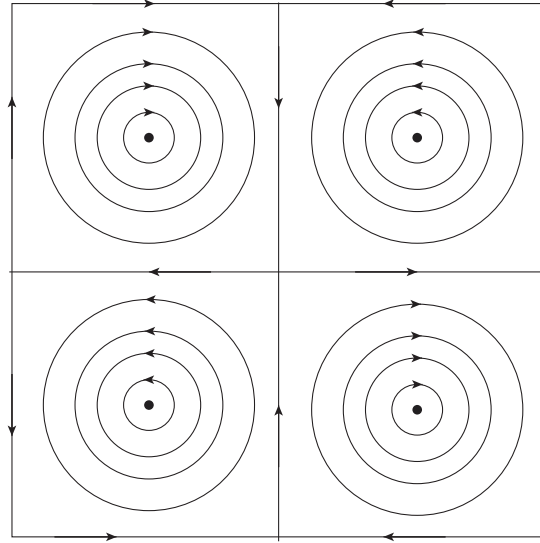

 FIGURE 8. Replacing a  $\partial$ -sink with a pair of a sink and a  $\partial$ -saddle

 FIGURE 9. The inverse operation of the Cherry blow-up operation  $\text{Ch}_l$  to a separatrix from a  $k$ -saddle to a sink


FIGURE 10. A flow on a torus with heteroclinic multi-saddle separatrices but without essential closed transversals nor essential periodic orbits

**Theorem 9.2.** *Each connected component of  $S - \text{Bd}$  for a quasi-regular flow  $v$  on a compact surface  $S$  is one of the following:*

- 1) *an open flow box in  $P$ ,*

- 2) an open annulus in  $P$ ,
- 3) a torus in  $\text{Per}(v)$ ,
- 4) a Klein bottle in  $\text{Per}(v)$ ,
- 5) an open annulus in  $\text{Per}(v)$ ,
- 6) an open Möbius band in  $\text{Per}(v)$ ,
- 7) an open essential subset in  $LD$ .

*Proof.* Replacing  $\partial_{\text{Per}}$  into centers, we may assume that  $\partial_{\text{Per}} = \emptyset$ . Since the boundary  $\partial S$  is contained in  $\text{Bd}$ , taking the double of a manifold, we may assume that  $S$  is closed. Fix a connected component  $U$  of  $S - \text{Bd}$ . Lemma 5.15 implies that the open subset  $U$  is contained in either  $\text{int Per}(v)$ ,  $\text{int}(P - P_{\text{sep}})$ , or  $LD$ . Suppose that  $U \subseteq LD$ . Since  $LD$  is open, Fix a point  $x \in U$  and a transverse arc  $I \subset U$  where  $x$  is the interior point in  $I$ . By the water fall construction, there is a closed transversal in  $U$  intersecting  $x$ . By Lemma 3.3, the closed transversal is essential and so is  $U$ . Suppose that  $U \subseteq \text{int Per}(v)$ . If  $\partial U = \emptyset$ , then  $U$  is a torus or a Klein bottle. Thus we may assume that  $\partial U \neq \emptyset$ . Then  $U$  is an open annulus or an open Möbius band. Suppose that  $U \subseteq \text{int}(P - P_{\text{sep}})$ . The  $\omega$ -limit (resp.  $\alpha$ -limit) set of each point in  $U$  is either an exceptional quasi-minimal set, a limit circuit or a sink (resp. a source). Cutting an essential periodic orbit and pasting two center disks (i.e. Taking operation  $C_o$ ), by induction, we may assume that each periodic orbits are null homotopic. Removing limit circuits, taking the metric completion, and collapsing the new boundaries into singletons, we may assume that there are no limit circuits. Collapsing the closures of multi-saddle separatrices into singletons, we may assume that there are no multi-saddle separatrices. Then the  $\omega$ -limit (resp.  $\alpha$ -limit) set of each point in  $U$  is either an exceptional quasi-minimal set or a sink (resp. a source). Suppose that  $\overline{U} \cap E = \emptyset$ . Note that the case  $U$  contained in the sphere satisfies the condition  $\overline{U} \cap E = \emptyset$ . Then we show that  $U$  is an open flow box in  $P$  by cutting  $U$  into small open flow boxes. Indeed, the  $\omega$ -limit (resp.  $\alpha$ -limit) set of each point in  $U$  is a sink (resp. a source). Fix a sink  $s$  which is the  $\omega$ -limit set of a point in  $U$  and a small closed transversal  $\gamma$  which is a boundary of a sink disk of the sink  $s$ . If  $\gamma$  is contained in  $U$ , then  $U$  is an open annulus in  $P$ . Thus we may assume that  $\gamma$  is not contained in  $U$ . Then there are separatrices  $O_i$  between the sink  $s$  and multi-saddles such that  $O_i \subset \partial U$ . Since each multi-saddle in  $\partial U$  connects between a sink and a source, the boundary  $\partial U$  consists of sinks, sources, multi-saddles, and separatrices of multi-saddles connecting sinks or sources. Taking a double of  $\partial U$ , the resulting flow on a closed surface consists of sinks, sources, 0-saddles, and non-closed proper orbits. Since the index of each singular point is non-negative, the existence of singular points and the Poincaré-Hopf theorem implies that  $S$  is a sphere and there are one sink and one source except 0-saddles on the double. This means  $U$  is an open flow box in  $P$ . Thus we may assume that  $\overline{U} \cap E \neq \emptyset$ . This implies that  $S$  is not spherical. Cutting an essential closed transversal in  $LD$  and pasting one sink disk and one source disk, we may assume that  $LD = \emptyset$ . Cutting an essential closed transversal  $\gamma_i$  intersecting  $E$  and pasting one sink disk and one source disk (i.e. applying an operation  $C_t$ ) as possible, let  $T$  the resulting surface from  $S$ ,  $w$  the resulting flow, and  $U^- \subseteq T$  be the resulting subset of  $U \setminus \bigcup_i \gamma_i$ . By Lemma 9.1, the resulting surface  $T$  is spherical. Let  $U_k$  be the connected components of  $U^-$  with  $U^- = \bigsqcup_k U_k$ . Denote by  $\tilde{U}$  (resp.  $\tilde{U}_k$ ) the saturation of  $U^-$  (resp.  $U_k$ ) by  $w$ . By the proof of spherical case, each connected component  $\tilde{U}_k$  of  $\tilde{U}$  is an open flow box such that  $\omega(x) = a_k$  and  $\alpha(x) = b_k$  for

some new sink  $a_k$  and some new source  $b_k$ . Then each connected component  $U_k$  of  $U \setminus \bigcup_i \gamma_i$  is an open flow box each of whose transverse boundary components is a closed interval in an essential closed transversal  $\gamma_i$ . In other words, each connected component  $U_k$  has one positive transverse boundary and one negative transverse boundary. Since  $U$  is constructed by pasting a positive boundary and a negative boundary of  $U_k$  as possible, we have  $U$  is an open flow box in  $P$ .  $\square$

The previous theorem implies the following statements.

**Corollary 9.3.** *Each connected component of  $S - \text{BD}$  for a quasi-regular flow  $v$  on a compact surface  $S$  is one of the following:*

- 1) *an open flow box in  $P$ ,*
- 2) *an open annulus in  $P$ ,*
- 3) *a torus in  $\text{Per}(v)$ ,*
- 4) *an open annulus in  $\text{Per}(v)$ ,*
- 5) *an open essential subset in  $\text{LD}$ .*

**Corollary 9.4.** *The following statement hold for a quasi-regular flow  $v$  on a compact surface  $S$ :*

- 1)  *$(S - \text{BD})/\hat{v}$  is a disjoint union of singletons, open intervals and circles.*
- 2) *If  $v$  is of finite type, then  $(S - \text{BD})/\hat{v}$  is a finite disjoint union of open intervals and circles.*
- 3)  *$\text{LD} = \emptyset$  if and only if  $(S - \text{BD})/v$  is a disjoint union of open intervals and circles.*

Recall that a quasi-regular flow without quasi-minimal sets (i.e.  $\text{LD} \sqcup \text{E} = \emptyset$ ) is of finite type if there are at most finitely many limit cycles. In the finite type, we obtain the following corollary.

**Theorem 9.5.** *The following statements hold for a flow  $v$  of finite type on a compact surface  $S$ :*

- 1) *The orbit space  $(S - D_{\text{ss}})/v$  (resp.  $(S - \text{Bd})/v$ ) is a finite disjoint union of intervals and circles.*
- 2) *The orbit space  $(S - \text{BD})/v$  is a finite disjoint union of open intervals and circles.*

## 10. GRAPHS OF SURFACE FLOWS

**10.1. Abstract multi-graphs of separatrices.** Lemma 6.2 and Corollary 8.3 imply the following statement.

**Corollary 10.1.** *The quotient space  $D_{\text{ss}}/v_{\text{ex}} = D_{\text{ss}}/\hat{v}_{\text{ex}}$  (resp.  $\text{Bd}/v_{\text{ex}} = \text{Bd}/\hat{v}_{\text{ex}}$ ) for a quasi-regular flow with  $\text{E} = \emptyset$  on a compact surface is an abstract multi-graph.*

Denote by  $G_{\text{ss}} = (V_{\text{ss}}, E_{\text{ss}})$  the abstract multi-graph  $D_{\text{ss}}/\hat{v}_{\text{ex}}$ . Note that each of  $V_{\text{ss}}$  and  $E_{\text{ss}}$  is a subset of  $G_{\text{ss}} = D_{\text{ss}}/\hat{v}_{\text{ex}}$ . For a quasi-regular flow  $v$  on a compact surface, define a label  $l_{E_{\text{ss}}} : E_{\text{ss}} \rightarrow 2^{D_{\text{ss}}/\hat{v}} \times 2^{D_{\text{ss}}/\hat{v}}$  by  $l_{E_{\text{ss}}}(O) := (\alpha(O)/\hat{v}, \omega(O)/\hat{v})$ . Recall that  $S_s \subseteq V_{\text{ss}}$  is the set of sinks,  $\partial$ -sinks, sources, and  $\partial$ -sources. Define a label  $l_{V_{\text{ss}}}(x)$  for a singular point  $x \in S_s$  as follows:

$l_{V_{\text{ss}}}(x)$  is the cyclic order  $<_c$  near  $x$  if  $x$  is either a sink or a source.

$l_{V_{\text{ss}}}(x)$  is the total order  $<_t$  near  $x$  if  $x$  is either a  $\partial$ -sink or a  $\partial$ -source.

Recall that the orbit space  $D/v$  for a quasi-regular flow on a compact surface is a finite abstract multi-graph. Define  $D_+ := D \cup \delta \text{Per}(v) \subseteq D_{\text{ss}}$ . Denote by

$G_{D_+} = (V_{D_+}, E_{D_+})$  the abstract multi-graph  $D_+/v$ . Define a label  $L_{D_+}(v)$  by the isotopy class of the inclusion of the generalized surface directed graph  $D_+$  on the surface  $S$ . Note that the set of isotopy classes of embeddings of realizations of finite abstract multi-graphs on a compact surface is enumerable (i.e. countable by an algorithm). Therefore the set of labels  $L_{D_+}$  for flows of finite type on compact surfaces is enumerable. Let  $\Gamma$  be the set of two-sided limit circuits with the collar directions (resp. one-sided limit circuits). Define a label  $l_{D_+}(\gamma)$  for a two-sided limit circuit with the collar direction by the cyclic order  $<_c$  near  $\gamma$ . If  $v$  is of finite type, then the union  $D_+$  can be reconstructed as both a topological space and a surface directed graph by the abstract multi-graph  $G_{D_+} = D_+/v$  with the label  $L_{D_+}$ .

**Lemma 10.2.** *Let  $I$  be the subgraph of  $G_{D_+}$  consisting of isolated vertices. We have  $(\delta \text{Per}(v))/v = I$ .*

**Lemma 10.3.** *Suppose that  $D$  has at most finitely many limit cycles. The following statements hold:*

- 1)  $G_{D_+} = D_+/v$  is a finite abstract multi-graph.
- 2)  $D_+$  is a disjoint union of a finite graph and finitely many circles.
- 3)  $(D_{ss} - D)/\hat{v}_{ex} = (D_{ss} - D)/\hat{v}$ .
- 4)  $(D_{ss} - D)/\hat{v}_{ex} = (D_{ss} - D)/v$  if  $E = \emptyset$ .
- 5)  $G_{ss} = D_{ss}/\hat{v}_{ex}$  is a finite abstract multi-graph if  $E \subseteq \min S$ .

Let  $S_s$  be the set of sinks,  $\partial$ -sinks, sources, and  $\partial$ -sources. In other words, the subset  $S_s$  is the set of singular points except centers and multi-saddles.

**Lemma 10.4.** *Let  $p : D_{ss} \rightarrow G_{ss}$  be the canonical projection. Suppose that  $v$  is quasi-regular,  $E = \emptyset$ , and  $D_+$  has at most finitely many limit cycles. The following statements hold:*

- 1)  $G_{ss} = D_{ss}/v_{ex}$  is a finite poset with respect to the specialization pre-order  $\leq_{ss}$  for the quotient topology.
- 2)  $V_{ss} = (D_+ \sqcup S_s)/v_{ex}$
- 3)  $E_{ss} = (D_{ss} - (D_+ \sqcup S_s))/v_{ex}$

By Lemma 10.3 and Lemma 10.4, for a quasi-regular flow with  $E = \emptyset$  on a compact surface, if  $D_+$  has at most finitely many limit cycles, then the ss-multi-saddle connection diagram  $D_{ss}$  can be reconstructed as both a topological space and a surface directed graph by the abstract multi-graphs  $G_{ss}$  and  $G_{D_+}$  with the labels  $l_{V_{ss}}$ ,  $l_{E_{ss}}$ , and  $L_{D_+}$ .

**10.2. Dual graphs of the ss-multi-saddle connection diagram.** Define the dual graph  $G^{ss} = (V^{ss}, E^{ss})$  of the ss-multi-saddle connection diagram  $D_{ss}$  as follows: The vertex set  $V^{ss}$  consists of the connected components of  $S - D_{ss}$ . An edge  $U_i U_j$  exists if and only if  $\dim(\overline{U_i} \cap \overline{U_j}) = 1$  and the closure  $\overline{U_i} \cap \overline{U_j}$  is not a union of limit circuits. Define the dual graph  $G^{Bd} = (V^{Bd}, E^{Bd})$  of the border point set  $Bd$  of a flow  $v$  on  $S$  in the same manner. By the construction, the abstract multi-graph  $G^{Bd}$  is isomorphic to the abstract multi-graph  $G^{ss}$  if a flow  $v$  on  $S$  is quasi-regular.

**10.3. Labels of  $V^{ss}$ .** For a quasi-regular flow  $v$  on a compact surface, define a label  $l^{ss} : V^{ss} \rightarrow \{\mathbb{D}, \mathbb{A}^+, \mathbb{A}^-, \mathbb{T}, \mathbb{K}, \mathbb{A}, \mathbb{M}, \mathbb{L}\}$  as follows: the label  $l^{ss}(U)$  is  $\mathbb{D}$  if  $U$  is an open flow box in  $P$ , the label  $l^{ss}(U)$  is  $\mathbb{A}^+$  (resp.  $\mathbb{A}^-$ ) if  $U$  is an open annulus in  $P$  but not (resp. and) a Reeb domain, and the label  $l^{ss}(U)$  is  $\mathbb{T}$  (resp.  $\mathbb{K}, \mathbb{A}, \mathbb{M}, \mathbb{L}$ )

if  $U$  is a torus in  $\text{Per}(v)$  (resp. a Klein bottle in  $\text{Per}(v)$ , an open annulus in  $\text{Per}(v)$ , an open Möbius band in  $\text{Per}(v)$ , an open essential subset in LD). We also define a label  $l^{D_{ss}} : V^{ss} \rightarrow 2^{D_{ss}/v} \cup \{\{H, K\} \mid H, K \subseteq D_{ss}/v\} \cup (2^{D_{ss}/v} \times 2^{D_{ss}/v})$  as follows:  $l^{D_{ss}}(U) := (\alpha(x)/v, \omega(x)/v)$  for some  $x \in U$  if  $l^{ss}(U) = \mathbb{D}, \mathbb{A}^\pm$ , or  $\mathbb{L}$  (i.e.  $U \subseteq P \sqcup \text{LD}$ ),  
 $l^{D_{ss}}(U) := \partial U/v$  if  $l^{ss}(U) = \mathbb{M}, \mathbb{T}$ , or  $\mathbb{K}$  (i.e.  $U$  is non-orientable or closed),  
 $l^{D_{ss}}(U) := \{\partial_- U/v, \partial_+ U/v\}$  if  $l^{ss}(U) = \mathbb{A}$ , where  $\partial_- U$  and  $\partial_+ U$  are the boundary components of the annulus  $U$ . By Theorem 9.4, the label  $l^{D_{ss}}$  is well-defined and is a pair of connected subsets. Note that  $l^{D_{ss}}(U) = (\emptyset, \emptyset)$  if and only if  $U$  is a closed surface. Moreover, define a label  $l^{V_{ss}} := (l^{ss}, l^{D_{ss}}) : V^{ss} \rightarrow \{\mathbb{D}, \mathbb{A}^+, \mathbb{A}^-, \mathbb{T}, \mathbb{K}, \mathbb{A}, \mathbb{M}, \mathbb{L}\} \times 2^{D_{ss}/v} \times 2^{D_{ss}/v}$  by  $l^{V_{ss}}(U) := (l^{ss}(U), l^{D_{ss}}(U))$

**10.4. Reductions.** Denote by  $\chi(S)$  the space of flows on a compact surface  $S$ . Put  $\chi := \bigcup \chi(T)$ , where the union runs over compact surfaces. Denote by  $\chi_F \subset \chi$  the subspace of flows of finite type (i.e. quasi-regular flows without quasi-minimal sets such that there are at most finitely many limit cycles). The class  $\chi_F$  contains some good class. In particular, we have  $\Sigma^r(S) \sqcup \mathcal{H}_*^r(S) \subset \chi_F$ , where  $\Sigma^r(S)$  is the subset of  $\chi^r(S)$  formed by the Morse-Smale  $C^r$ -vector fields and  $\mathcal{H}_*^r(S)$  is the set of regular Hamiltonian  $C^r$  vector field each of whose saddle connection is self-connected. Then a flow  $v$  in  $\chi_F$  can be reconstructed by the finite data  $(G_{ss}, l_{V_{ss}}, l_{E_{ss}}, G_{D_+}, l_{D_+}, L_{D_+}, G^{ss}, l^{V_{ss}})$ . To state precisely, we define some notations as follows: Denote by  $\mathcal{G}_l$  the set of finite abstract multi-graphs with labels. Define a mapping  $p : \chi_F \rightarrow \mathcal{G}_l \times \mathcal{G}_l \times \mathcal{G}_l$  by

$$p(v) := (G_{ss}, l_{V_{ss}}, l_{E_{ss}}, G_{D_+}, l_{D_+}, L_{D_+}, G^{ss}, l^{V_{ss}})$$

and an equivalent relation  $\sim$  on  $\chi_F$  by  $v \sim w$  if either there is a homeomorphism  $h : S \rightarrow T$  which is orbit-preserving (i.e. the image of an orbit of  $v$  is an orbit of  $w$ ) and preserves orientation of the orbits, or there is an orbit-preserving homeomorphism  $h : S \rightarrow T$  which reverses orientation of the orbits. Write  $\widetilde{\chi}_F$  by the quotient space of  $\chi_F$  by the equivalence  $\sim$ . Now we can state precisely that a flow  $v$  in  $\chi_F$  can be reconstructed by the finite data as follows.

**Theorem 10.5.** *The induced map  $\tilde{p} : \widetilde{\chi}_F \rightarrow \mathcal{G}_l \times \mathcal{G}_l \times \mathcal{G}_l$  is well-defined and injective.*

*Proof.* First, the union  $D_+ = D \cup \delta \text{Per}(v) \subseteq D_{ss}$  can be reconstructed as both a topological space and a surface directed graph by the abstract multi-graph  $G_{D_+} = D_+/v$  with the label  $L_{D_+}$  which is the isotopy class of the inclusion of  $D_+$  on the surface  $S$ . Second, the ss-multi-saddle connection diagram  $D_{ss}$  can be reconstructed as both a topological space and a “surface directed graph” by the union  $D_+$  and by the abstract multi-graph  $G_{ss}$  with the labels  $l_{V_{ss}}$  and  $l_{E_{ss}}$ . Third, each connected component of the complement of  $D_{ss}$  can be reconstructed by the vertex set  $V^{ss}$  of  $G^{ss}$  and by the label  $l^{ss}$ . Finally, we can paste  $D_{ss}$  and the complement by the edge set  $E^{ss}$  of  $G^{ss}$  and the label  $l^{D_{ss}}$ .  $\square$

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